Anyon crystals from statistics

Domenico Orlando
INFN | Torino

Symmetry Seminar | 15 October 2026

arXiv:2609.25249 with Zohar Komargodski, Xuzixiang Lou, Ivri Nagar, Susanne Reffert

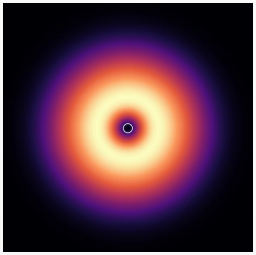


THE MESSAGE

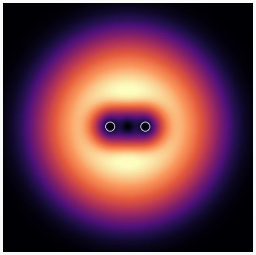
A free Hamiltonian with non-trivial statistics can have a crystaline ground state



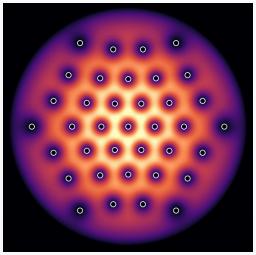
(a) $\lambda = 1.5$
Chiral droplet



(b) $\lambda = 4.5$
Centered unit vortex



(c) $\lambda = 8$
Split multivortex



(d) $\lambda = 37$
Triangular vortex crystal

FREE ANYONS: BOUNDARY CONDITIONS

$$H = \sum_{a=1}^{N} \left(\frac{p_a^2}{2} + \frac{\omega^2 r_a^2}{2} \right)$$

with a twist

$$\Psi(\dots, x_i, \dots, x_j, \dots) = e^{i\theta} \Psi(\dots, x_j, \dots, x_i, \dots)$$

in the following $\theta = \pi/k$.

- for $k = 1$, $e^{i\theta} = -1$: spinless fermions
- for $k \to \infty$, $e^{i\theta} = 1$: free bosons

The harmonic potential is particularly nice because of the state/operator correspondence.
Any confining potential would do.



FREE ANYONS: GAUGE CONNECTION

A singular **gauge transformation** changes the BC (now periodic)

$$\Psi_A(t, z_a) = \exp\left[\frac{i}{k}\sum_{a<b}(z_a - z_b)\right]\Psi_B(t, z_a)$$

The kinetic term becomes

$$\frac{p_a^2}{2} \to \frac{(p_a + a_a)^2}{2} \qquad \nabla \times a = \frac{2\pi}{k}n(x)$$

We trade the BC for a magnetic flux tube attached to each particle

$$\oint_{\text{particle}} a \cdot d\ell = \frac{2\pi}{k}$$

The anyon statistics is written as a **Aharonov–Bohm phase**. (I still need to tell you the coincident-point BC).



(FAKE) FIELD THEORY: CHERN-SIMONS

Rewrite in Langrangian formalism (but it's still **QM in a trench coat**)

$$L = i\Phi^* D_t \Phi - \frac{1}{2}|D_i \Phi|^2 - \frac{\omega^2 r^2}{2}|\Phi|^2 - \frac{\pi}{k}|\Phi|^4 + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

- The quartic term implements the coincident-point BC (more on that later).
- The CS term is needed so that the EOM for A_0 implements the Gauss law:

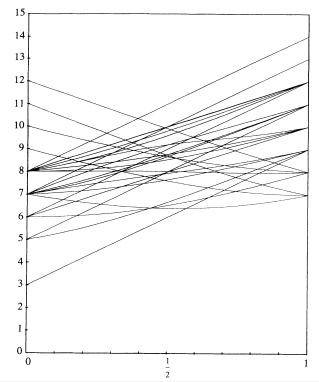
$$B = \frac{2\pi}{k}|\Phi|^2 \qquad\qquad |\Phi(x)|^2 = n(x)$$

No external field (just the harmonic trap).



STRUCTURE OF THE GROUND STATE

The structure of the ground state as a function of k is known to be complicated because states keep crossing.



Exact spectrum for 4 anyons in a harmonic potential

[Sporre, Verbaarschot, Zahed]

The number of crossings grows with the number of particles.

WHAT DOES THE GROUND STATE LOOK LIKE

- **Near the fermion point** (for $k = 1 + \varepsilon$) it's an **unconvential superfluid**. There are no Cooper pairs. Fermions have a small magnetic flux attached. No Fermi surface, but a Hall response that cancels the statistical Chern–Simons term. The remaining Maxwell mode describes a superfluid [LATE 80ES].
- Recently [DU, MEHTA, SON] showed that there are **roton modes**. Instability?
- **Near the bosonic point**, mean-field analysis has shown the presence of **vortices** [CORREGGI, DUBOSCQ, LUNDHOLM, ROUGERIE].

What is the physics of the ground state along the line $k : 1 \to \infty$?



TODAY'S TALK: CONCLUSIONS

Ground state



TODAY'S TALK: CONCLUSIONS

Ground state

- Can the statistics alone crystallize a gas?
- Is the vortex lattice superfluid?



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

- What is the fine structure of the ground state when we vary k?



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

What are the fixed points?



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

What are the fixed points?

- I used a precise value of the $|\Phi|^4$ coupling (repulsive). Are there others?



TODAY'S TALK: CONCLUSIONS

Ground state

- The ground state is a triangular vortex lattice ordinary solid

Level crossing

What are the fixed points?



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

- The rotor modes form an Anderson tower and we get $\mathcal{O}(N^2)$ crossings

What are the fixed points?



TODAY'S TALK: CONCLUSIONS

Ground state

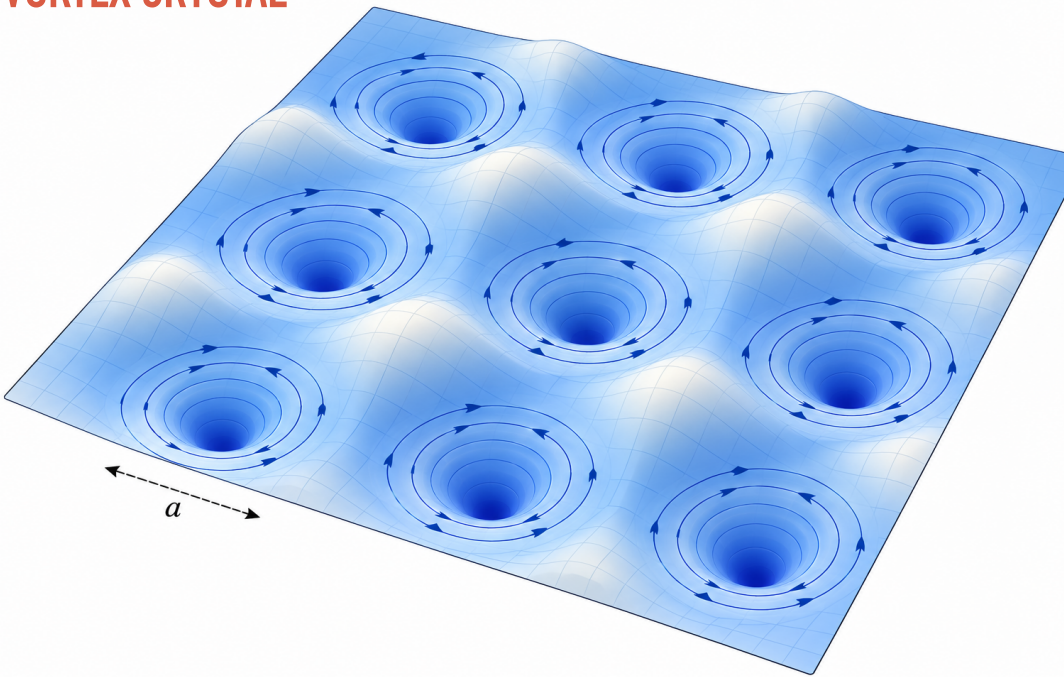
Level crossing

What are the fixed points?

- The attractive-interaction fixed point seems sick
- Possible fixed point web corresponding to higher-order boundary conditions



THE VORTEX CRYSTAL



LARGE CHARGE DOUBLE SCALING

$$S = \int dt\, d^2x \left[i\Phi^* D_t \Phi - \frac{1}{2}|D_i \Phi|^2 - \frac{\omega^2 r^2}{2}|\Phi|^2 - \frac{\pi}{k}|\Phi|^4 + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right]$$

with the normalization

$$\int d^2x\, |\Phi|^2 = N$$

Redefine $\Phi(t, x) = \sqrt{k}\sqrt{\rho(t, x)}e^{-i\chi(t,x)}$ and take the double scaling limit

$$N \to \infty, \qquad k \to \infty, \qquad \frac{N}{k} = \lambda \text{ fixed}$$

The interaction per particle is small, but the **collective statistical flux is finite**.



THE DOUBLE-SCALED ACTION

$$N \to \infty, \qquad k \to \infty, \qquad \frac{N}{k} = \lambda \text{ fixed}$$

The action becomes

$$S = k \int dt\, d^2x \left[\rho(\dot{\chi} - A_0) - \frac{1}{4\pi}\varepsilon^{ij} A_i \dot{A}_j + \frac{A_0}{2\pi} B - \frac{(\nabla\rho)^2}{8\rho} - \frac{\rho}{2}(\nabla\chi - A)^2 - \pi\rho^2 - \frac{\omega^2 r^2}{2}\rho \right]$$

with the normalization

$$\int d^2x\, \rho = \lambda$$

New **semiclassical** description $\hbar = \frac{1}{k}$ **controls the quantum fluctuations**.
Not the usual weak coupling for fixed λ.



A CONSTRAINED HAMILTONIAN

We have reduced our problem to an optimization

$$\frac{H}{k} = \int d^2x \left[\frac{(\nabla\rho)^2}{8\rho} + \frac{\rho}{2}(\nabla\chi - A)^2 + \pi\rho^2 + \frac{r^2}{2}\rho \right]$$

with

$$\nabla \times A = 2\pi\rho \qquad \int d^2x\, \rho = \lambda$$

- $(\nabla\rho)^2/8\rho$ is the "quantum pressure" (very much leading order in our limit)
- $\rho(\nabla\chi - A)^2/2$ is the gauge-invariant kinetic energy
- $\pi\rho^2$ is the contact interaction
- $r^2\rho/2$ is the harmonic trap



THE VORTICES START APPEARING

To minimize the kinetic energy we would like to have

$$v = \nabla\chi - A = 0$$

Since $\nabla \times A = 2\pi\rho$, there is no smooth χ that would give $v = 0$.

The situation changes if we put **vortices**; then:

$$\oint \nabla\chi \cdot d\ell = 2\pi w \Rightarrow \oint_{\partial\Sigma} v \cdot d\ell = 2\pi \left(N_v - \frac{N_\Sigma}{k} \right)$$

so we expect a screening for

$$N_v \sim \frac{N}{k}$$

as long as vortives are not too expensive.



THE VORTEX-FREE (CHIRAL) SOLUTION

Without vortices, solve the Gauss constraint in Coulomb gauge to get the Taubes equation

$$\nabla^2 \log(\rho) = 4\pi\rho - 4$$

Qualitatively:

- at small λ it's a **Gaussian droplet** (similar to the free boson)

$$\rho(r) \sim \frac{\lambda}{\pi} e^{-r^2}$$

- at large λ is **flat** with an edge layer

$$\rho(r) \sim \frac{1}{\pi} \Theta(\sqrt{\lambda} - r)$$



THE VORTEX-FREE (CHIRAL) SOLUTION

It reproduces an exact quantum state (the repulsive version of the famous Jackiw-Pi solution)

$$\Psi \sim e^{-1/2 \sum_i |z_i|^2} \prod_{i<j} (z_i - z_j)^{1/k}$$

It **cannot be a ground state for large λ**: the energy is

$$\frac{E(\lambda)}{k} = \frac{\lambda^2}{2} + \lambda$$

and from general principles at large λ in a harmonic potential **we must have** $E(\lambda) \sim \lambda^{3/2}$.



ONE VORTEX

What does a vortex look like?

$$\chi = \ell\varphi, \qquad \rho = q(r)^2, \qquad A_\varphi = \frac{Q(r)}{r}$$

where $Q(r)$ is the charge enclosed in a radius r

$$Q(r) = 2\pi \int_0^r s\, ds\, \rho(s)$$

and the energy becomes

$$\frac{E_\ell}{k} = 2\pi \int_0^r r\, dr \left[\frac{1}{2} \left(\frac{dq}{dr} \right)^2 + \frac{q^2 (\ell - Q)^2}{2r^2} + \pi q^4 + \frac{r^2 q^2}{2} \right]$$

where we see the competition $\ell - Q(r)$.

We minimize this object numerically and find that at $\bar{\lambda} \sim 2.96$, $E_0(\bar{\lambda}) = E_1(\bar{\lambda})$.

One vortex has appeared.



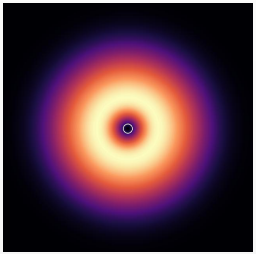
AND THEN MANY MORE

What then, higher ℓ? Multiple-wound vortices are **unstable and split**.

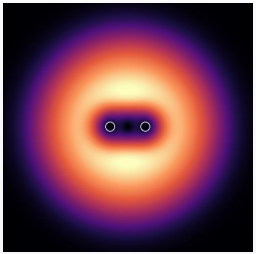
From here on one proceeds numerically



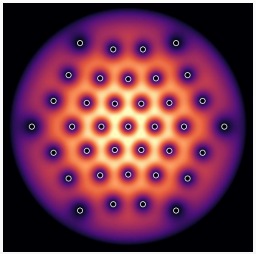
(a) λ = 1.5
Chiral droplet



(b) λ = 4.5
Centered unit vortex



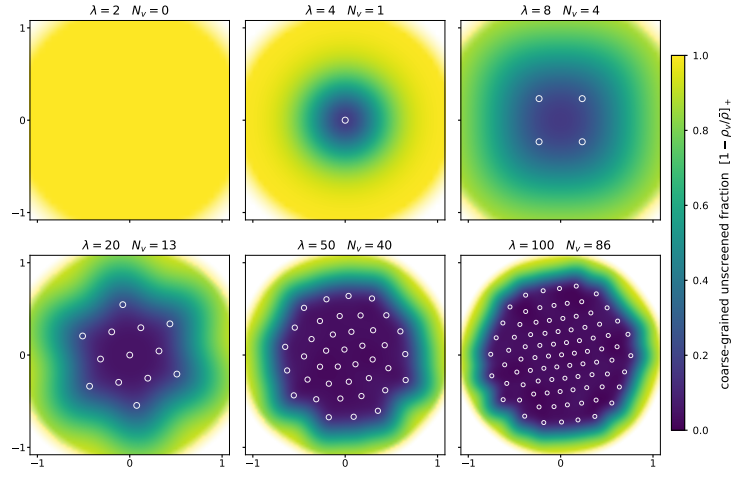
(c) λ = 8
Split multivortex



(d) λ = 37
Triangular vortex crystal

DYNAMICAL BULK SCREENING

For larger and larger λ, the statistical magnetic field is screened

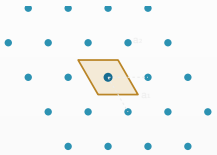


$$1 - \frac{\rho_v}{\rho} \sim 0$$

Each vortex contains exactly k anyons.

Going around a vortex, the overall phase vanishes.

TRIANGULAR LATTICE



$$\bar{\rho} = \frac{1}{A_{cell}} \int_{cell} d^2x\, \rho$$

For **perfect screening** there are **k anyons per cell**. The integral over the cell is 2π, and using $B/(2\pi) = \rho = n/k$

$$2\pi = \int_{cell} d^2x\, B = 2\pi \bar{\rho} A_{cell} \Rightarrow A_{cell} = \frac{k}{n}$$

Schrödinger invariance (d = 2, z = 2) fixes the equation of state

$$\frac{\mathcal{E}}{k} = b_e \bar{\rho}^2$$

and this number can be computed numerically:

$$b_2 = \frac{1}{A_{cell}\bar{\rho}^2} \min_{cell} \int_{cell} d^2x \left[\frac{(\nabla\rho)^2}{8\rho} + \frac{\rho v^2}{2} + \pi\rho^2 \right] = 7.233\ldots$$

THE THOMAS–FERMI PROFILE

With the harmonic trap, $\bar{\rho}$ is not constant, but we assume **slowly-varying (LDA)**.

$$\frac{E}{k} = \int d^2x \left[b_e \bar{\rho}^2 + \frac{\omega^2 r^2}{2} \bar{\rho} \right], \qquad \int d^2x \, \bar{\rho} = \lambda$$

standard **Thomas-Fermi profile** with the renormalized coefficient

$$\bar{\rho}(r) = \frac{1}{b_e} \left(\sqrt{\frac{2 b_e \lambda}{\pi}} - \frac{\omega^2 r^2}{2} \right) \Theta \left(\left(\frac{8 b_e \lambda}{\pi} \right)^{1/4} - \omega r \right)$$

and the energy is

$$\frac{E}{\omega} = \frac{2k}{3} \sqrt{\frac{2 b_e}{\pi}} \lambda^{3/2} \sim 1.43 \frac{N^{3/2}}{\sqrt{k}}$$

The $N^{3/2}$ is again from **conformal invariance**.



ENERGY COMPARISON

At $k = 1 + \varepsilon$ define $1/\tilde{k} = 1 - 1/k$. The **superfluid** has energy

$$E_{hom} = \frac{2\omega\sqrt{3}}{3}\sqrt{1 + \frac{\{\tilde{k}\}(1 - \{\tilde{k}\})}{\tilde{k}^2}}N^{3/2}$$

$\{\tilde{k}\}$ is the fractional part.

Compare with our

$$E_{crystal} \sim 1.43\frac{N^{3/2}}{\sqrt{k}}$$

We find that

$$E_{hom} = E_{crystal} \qquad \text{at } k = k_c \text{ between 2 and 3}$$



A CONVENTIONAL SOLID

We don't have vortices superimposed to a superfluid.

The single anyons can move (microscopic), but there is **no gapless collective mode** (macroscopic).

Call $u(t, x)$ the displacement field of the lattice.

Complete screening now locks also the fluctuations of the matter density.

$$\delta\bar{\rho} = \delta\rho_v = -\bar{\rho}\nabla \cdot u$$

so there is no independent matter fluctuation.

(The argument that $\chi \to \chi + \lambda$ can be reabsorbed by the gauge field is not sufficient: it would also apply to $k = 1 + \varepsilon$, where we know that there is an unconventional superfluid).



A CONVENTIONAL SOLID

The **phonons are standard**

$$u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \qquad u^T_{ij} = u_{ij} - \frac{1}{2}\delta_{ij} u_{kk}$$

and the action depends on two parameters K and G

$$L = \frac{k\bar{\rho}}{2}\int d^2x \left(\dot{u}^2 - \bar{\rho}\left(K(\nabla\cdot u)^2 + 2G u^T_{ij} u^T_{ij}\right)\right)$$

which fix the small-q dispersions

$$\omega_L(q) = \sqrt{\bar{\rho}(K+G)}\,q, \qquad \omega_T(q) = \sqrt{\bar{\rho}G}\,q$$

(we are in the LDA) Numerically $K = 14.446(8)$ and $G = 1.354(4)$.



MELTING

Lindemann criterion. The crystal melts if **quantum fluctuations are of the order of the distance** between the vortices

$$\gamma_L^2 = \frac{\langle u^2 \rangle}{a_v^2} \sim 0.25$$

With the two phonons

$$\gamma_L^2 = \frac{\sqrt{3}}{2\bar{\rho}} \sum_{s=L,T} \int_{BZ} \frac{d^2x}{(2\pi)^2} \frac{1}{2k\bar{\rho}\omega_s(q)} = \frac{\sqrt{3}}{4\sqrt{\pi}k} \left(\frac{1}{\sqrt{K+G}} + \frac{1}{\sqrt{G}} \right)$$

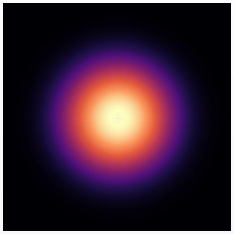
and we get melting at

$k = k_c$ between 3 and 5

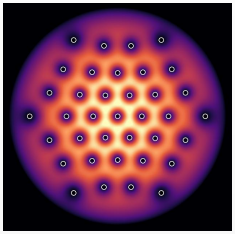


ZERO-TEMPERATURE PHASE DIAGRAM

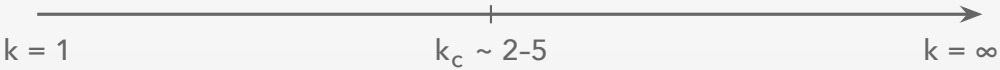
Forget for a moment that we are in the double-scaling limit.



Anyon superfluid

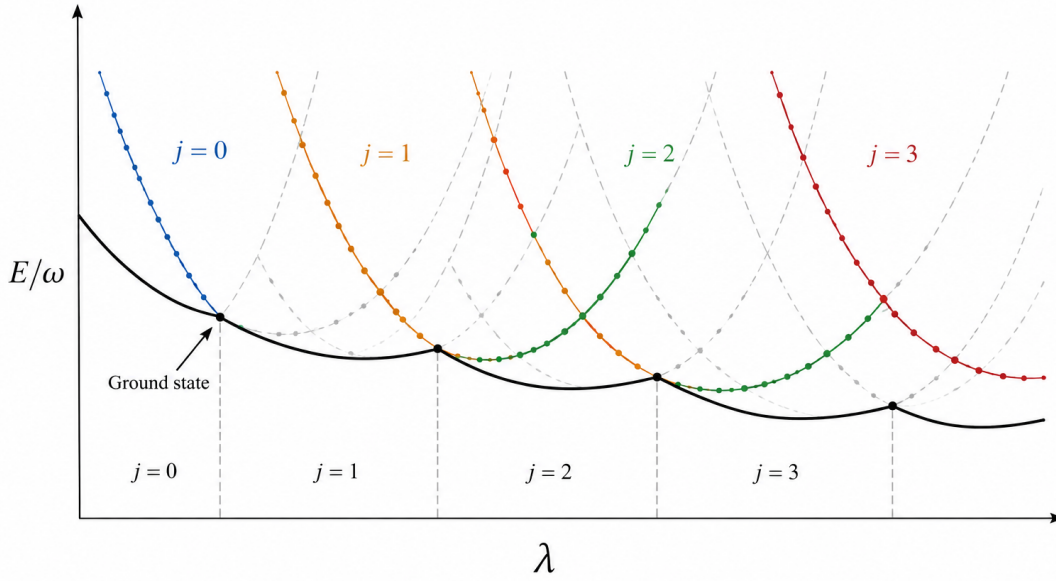


Vortex crystal



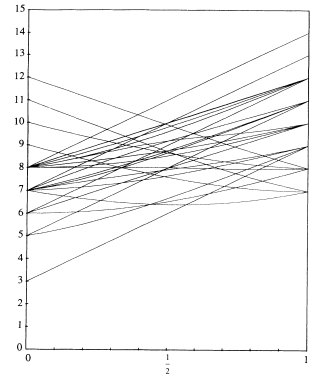
Preliminary Monte Carlo evidence

CROSSINGS



STRUCTURE OF THE GROUND STATE

The structure of the ground state as a function of k is known to be complicated because states keep crossing.



Exact spectrum for 4 anyons in a harmonic potential

[Sporre, Verbaarschot, Zahed]

The number of crossings grows with the number of particles.

Can we describe this phenomenon in our semiclassical story?

QUANTIZATION OF THE ZERO MODES

Remember

$$\Psi_A = \exp\left[\frac{i}{k}\sum_{a<b}(z_a - z_b)\right]\Psi_B$$

There are two contributions to J: an integer j from Ψ_B and the **statistical** piece:

$$J = j + \frac{N\lambda}{2}$$

The idea is that changing λ changes the value of j at the energy minimum.



SEMICLASSICAL PICTURE

The saddle **breaks rotational invariance**. At finite N there is an Anderson tower of rotor modes with angular momentum j.

In the semiclassical story, the saddle has **angular momentum**

$$J_{cl} = N\left(-\frac{1}{\lambda}\int d^2x\,\rho\,\partial_\varphi\chi + \frac{\lambda}{2}\right) = N\left(W(\lambda) + \frac{\lambda}{2}\right)$$

$W(\lambda)$ is the average phase winding.

J_{cl} is **not an integer** (not an eigenstate of angular momentum): we need **rotor states**.

For a slow rotation Ω of the saddle $J = J_{cl} + I\Omega$, and $E = E_{cl} + I\Omega^2/2$, where $I = \mathcal{O}(N)$.

$$E_j(\lambda) = E_{cl}(\lambda) + \frac{(j - NW(\lambda))^2}{2I}$$



THE CROSSINGS

$$E_j(\lambda) = E_{cl}(\lambda) + \frac{(j - NW(\lambda))^2}{2I}$$

Varying λ moves the **center of the rotor tower**.
Two sectors cross when

$$NW(\lambda_n) = n + \frac{1}{2}, \qquad n \in \mathbb{Z}$$

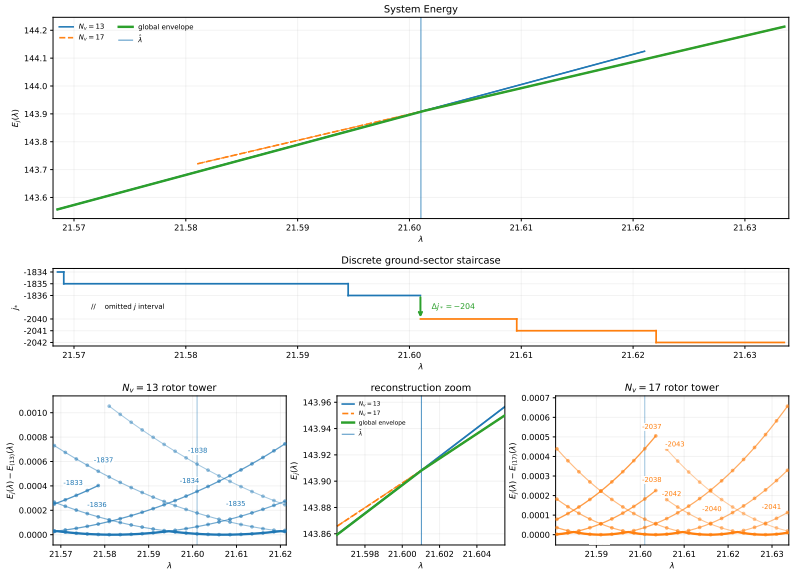
The crossing density is

$$\frac{dn_{cross}}{d\lambda} = N\left|\frac{dW}{d\lambda}\right| = \frac{N}{2}$$

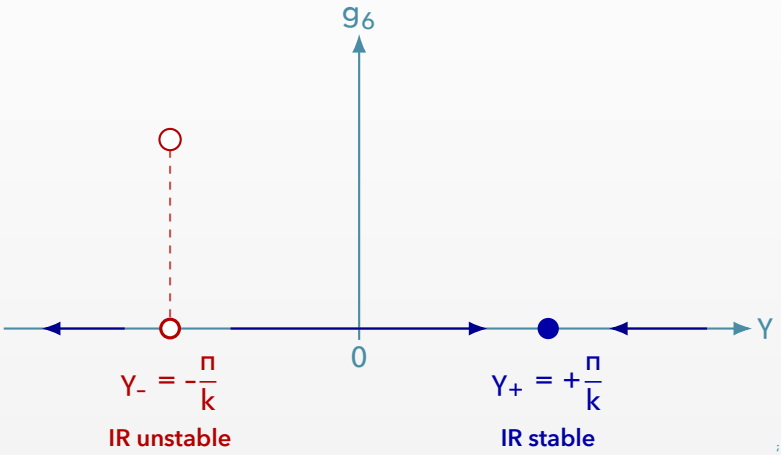
That there are $\mathcal{O}(N)$ crossings at fixed λ.
The changes of ground states are subdominant $\mathcal{O}(N^0)$



THE CROSSINGS



FIXED POINTS



HAMILTONIAN AND BC

Take two anyons. The exchange statistics is written for the relative cordinate as

$$\psi(r, \varphi + \pi) = e^{i\pi/k}\psi(r, \varphi)$$

The Hamiltonian is just free, but ψ is multivalued

$$\ell = 2n + \frac{1}{k}$$

So the radial Hamiltonian for $n = 0$ still gets a centrifugal term

$$H = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{k^2 r^2}$$

We need BC at $r = 0$ (coincidence). Two selfadjoint extensions:

$$\psi_+ \sim r^{1/k} \qquad \psi_- \sim r^{-1/k}$$



FIXED POINTS FROM THE HAMILTONIAN

Add a point-like interaction $V = g\delta^2(r)$.
g has an **anomalous dimension** and its beta function is

$$\beta_g = \frac{1}{2\pi}\left(g - \frac{2\pi}{k}\right)\left(g + \frac{2\pi}{k}\right)$$

There are no approximations. This is the **exact RG flow**.

- The repulsive interaction is RG stable
- The attractive interaction is RG unstable (Feshbach).

In field theory language, this is a quartic coupling $\pm\pi/k$

$$L = i\Phi^* D_t \Phi - \frac{1}{2}|D_i \Phi|^2 - \frac{\omega^2 r^2}{2}|\Phi|^2 \pm \frac{\pi}{k}|\Phi|^4 + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$



THE ATTRACTIVE-INTERACTION FIXED POINT

The slope of the β function gives the **dimension of the quartic operator**

$$\Delta[|\Phi|^4] = 4 \pm \frac{2}{k}$$

for repulsive/attractive.

So far, so good, but for N particles one gets

$$\Delta[|\Phi|^{2N}] = N \pm \frac{2}{k} N(N-1)$$

in the **attractive** case (–), at some point the operator becomes **relevant and violates the unitarity bound**.

- In **Hamiltonian** language, the BC for two coinciding particle is not enough to uniquely describe the interaction for three. More BC are needed.
- In Lagrangian language, we need to add $|\Phi|^6$, that will have its fixed point. And then possibly $|\Phi|^8$ and so on.



TODAY'S TALK: CONCLUSIONS

Ground state



TODAY'S TALK: CONCLUSIONS

Ground state

- The ground state is a triangular vortex lattice ordinary solid



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

- The rotor modes form an Anderson tower and we get $\mathcal{O}(N^2)$ crossings



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

What are the fixed points?



TODAY'S TALK: CONCLUSIONS

Ground state

Level crossing

What are the fixed points?

- The attractive-interaction fixed point seems sick
- Possible fixed point web corresponding to higher-order boundary conditions

