The Resurgence of the Large Charge expansion

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arXiv:2102.12488, arXiv:2311.14793, and arXiv:2505.21631

WHO'S WHO

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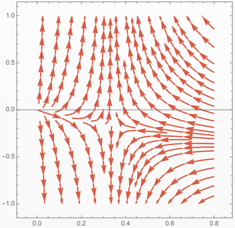
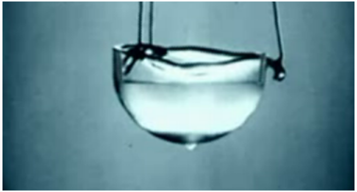
WHY ARE WE HERE? CONFORMAL FIELD THEORIES

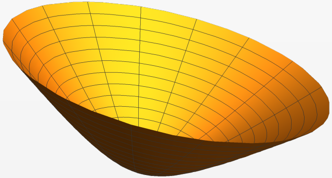
extrema of the RG flow



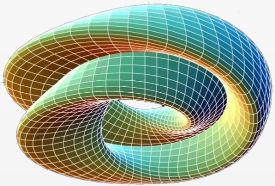
critical phenomena



quantum gravity



string theory



WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

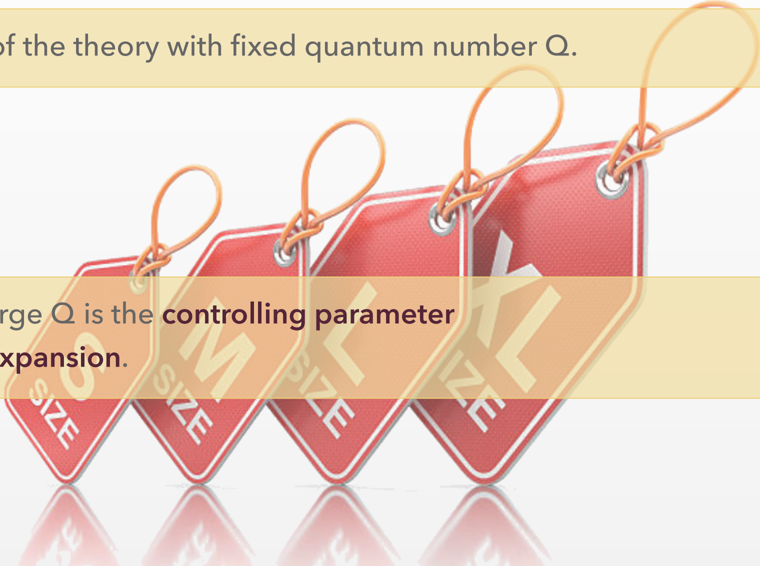
No parameter of the theory can be dialed to a simplifying limit.

WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

In presence of a **symmetry** there can be **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

THE IDEA

Study **subsectors** of the theory with fixed quantum number Q.



In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.

NOT AN ORIGINAL IDEA

NO BOOTSTRAP HERE!

This approach is orthogonal to bootstrap.

We will use an effective action.

We will access sectors that are difficult to reach with bootstrap.

(However, arXiv:1710.11161).

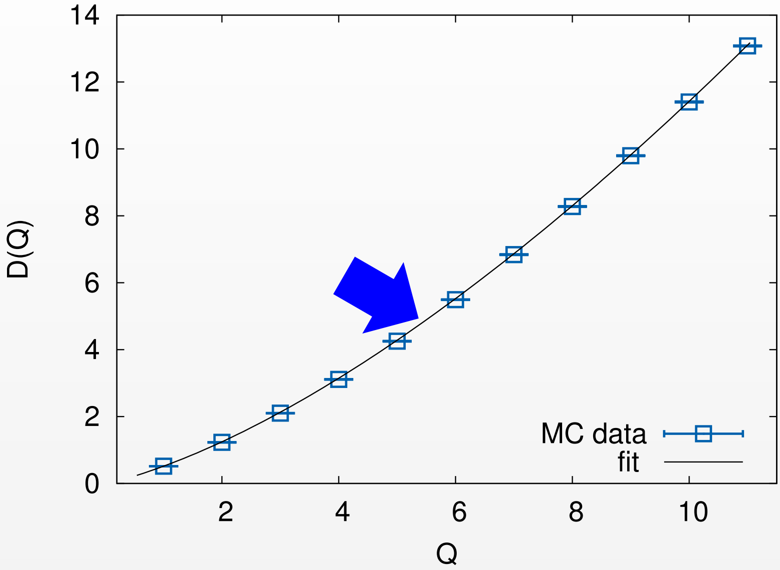
CONCLUSIONS

We consider the O(N) **vector model in three dimensions**. In the IR it flows to a **conformal fixed point** [Wilson & Fisher].

We find an explicit formula for the **dimension of the lowest primary at fixed charge**:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

CONCLUSIONS: O(2)



SCALES

We want to write a **Wilsonian effective action.**

Choose a cutoff Λ, separate the fields into high and low frequency φ_H, φ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\varphi_L)} = \int \mathcal{D}\varphi_H \, e^{iS(\varphi_H, \varphi_L)}$$

SCALES

We want to write a **Wilsonian effective action.**

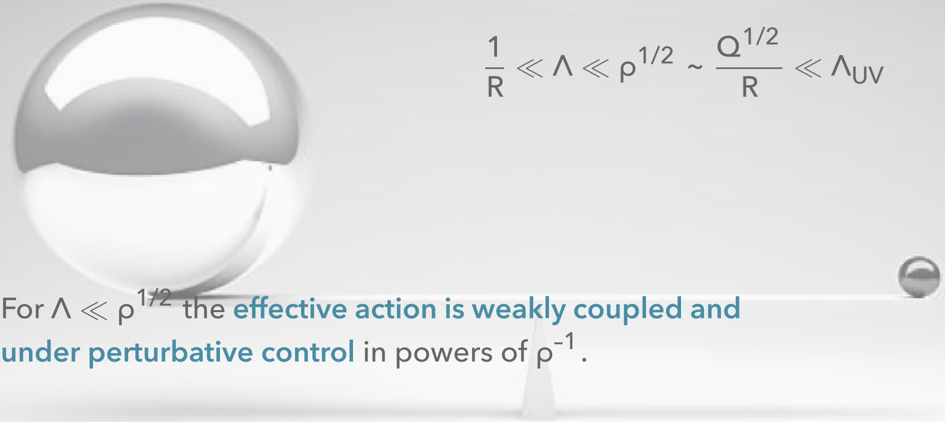
Choose a cutoff Λ, separate the fields into high and low frequency φ_H, φ_L and do the path integral over the high-frequency part:

too hard

$$e^{iS_\Lambda(\varphi_L)} = \int \mathcal{D}\varphi_H \, e^{iS(\varphi_H,\varphi_L)}$$

SCALES

- We look at a finite box of typical **length R**
- The U(1) charge Q fixes a **second scale $\rho^{1/2} \sim Q^{1/2}/R$**



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the **effective action is weakly coupled and under perturbative control** in powers of ρ^{-1}.

NON-LINEAR SIGMA MODEL

In a generic theory™, picking the lowest state of fixed charge induces a spontaneous symmetry breaking.
The low-energy physics is described by a **Goldstone field** χ.

Using conformal invariance, the most general action must take the form

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \ldots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$. All other terms are suppressed by powers of $1/Q$.

NON-LINEAR SIGMA MODEL

The **energy of the lowest state** for this action has the form

$$E = \frac{c_{3/2}}{\sqrt{V}} Q^{3/2} + R c_{1/2} \sqrt{V} Q^{1/2} + \ldots$$

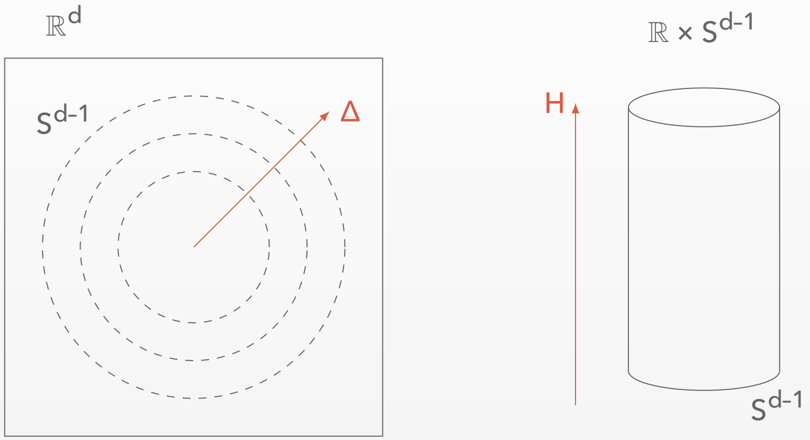
The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}} \zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

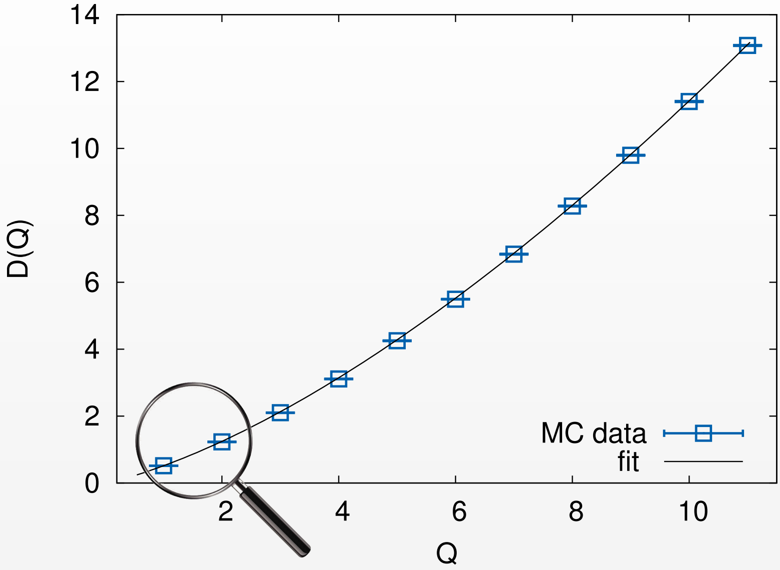
STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

TOO GOOD TO BE TRUE?



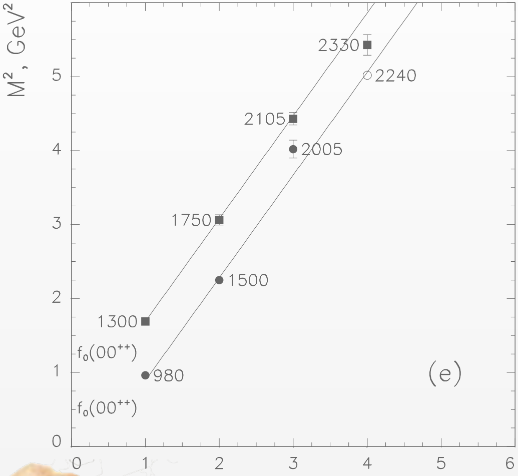
TOO GOOD TO BE TRUE?

Think of **Regge trajectories**.

The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but experimentally everything works so well at small J that String Theory was invented.



SELECTED TOPICS IN THE LARGE CHARGE EXPANSION

- **O(2) model** [Hellerman, DO, Reffert, Watanabe] [Monin, Pirtskhalava, Rattazzi, Seibold]
- **fermions** [Komargodski, Mezei, Pal, Raviv-Moshe] [Antipin, Bersini, Panopoulos] [Hellerman, Dondi, Kalogerakis, Moser, DO, Reffert]
- **holography** [Nakayama] [Loukas, DO, Reffert, Sarkar] [de la Fuente] [Guo, Liu, Lu, Pang] [Giombi, Komatsu, Offertaler]
- **large N** [Álvarez-Gaumé, DO, Reffert] [Giombi, Hyman]
- **ε double-scaling** [Badel, Cuomo, Monin, Rattazzi] [Arias-Tamargo, Rodriguez-Gomez, Russo] [Antipin, Bersini, Sannino, Wang, Zhang] [Jack, Jones]
- **non-relativistic CFTs** [Kravec, Pal] [Helllerman, Swanson] [Favrod, DO, Reffert] [DO, Reffert, Pellizzani] [Hellerman, DO, Reffert, Pellizzani, Swanson]
- **$\mathcal{N} = 2$** [Hellerman, Maeda] [Hellerman, Maeda, DO, Reffert, Watanabe] [Bourget, Rodriguez-Gomez, Russo] [Grassi, Komargodski, Tizzano] [Cremonesi, Lanza, Martucci]
- **bootstrap** [Jafferis, Zhiboedov]
- **resurgence** [Dondi, Kalogerakis, DO, Reffert] [Antipin, Bersini, Sannino, Torres] [Watanabe]

TODAY'S TALK

Justify and prove all these claims from first principles

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Use resurgence for the large-charge EFT

TODAY'S TALK

Justify and prove all these claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

Use resurgence for the large-charge EFT

TODAY'S TALK

Justify and prove all these claims from first principles

Use resurgence for the large-charge EFT

- Borel resum the double-scaling $Q \to \infty$, $N \to \infty$ limit
- geometric interpretation of non-perturbative effects
- general structure of the corrections in the EFT

PARENTAL

ADVISORY

EXPLICIT CONTENT

LARGE N VS. LARGE CHARGE

THE MODEL

φ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int dt\, d\Sigma \left[g^{\mu\nu} \left(\partial_\mu \varphi_i\right)^* \left(\partial_\nu \varphi_i\right) + r \varphi_i^* \varphi_i + \frac{u}{2} \left(\varphi_i^* \varphi_i\right)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.

We compute the partition function at fixed charge

$$Z(Q_1, \ldots, Q_N) = \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int d\Sigma\, j_i^0 = i \int d\Sigma \left[\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i \right].$$

FIX THE CHARGE

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i}\, \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i\, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i\, e^{-S^\theta[\varphi]}$$

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EFFECTIVE ACTION: COVARIANT DERIVATIVE

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int dt\, d\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u (\varphi_i^* \varphi_i)^2 \right)$$

$$\begin{cases} D_0 \varphi = \partial_0 \varphi + i \frac{\theta}{\beta} \varphi \\ D_i \varphi = \partial_i \varphi \end{cases}$$

Stratonovich transformation

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int dt\, d\Sigma \left[\left(D^i_\mu \varphi_i \right)^* \left(D^i_\mu \varphi_i \right) + (r + \lambda) \varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = m^2 + \hat{\lambda}$$

EFFECTIVE ACTION FOR $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_\theta[\hat{\lambda}] = \sum_{i=1}^{N} \left[V\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 + \hat{\lambda} \right) \right] \right].$$

Non-local action for $\hat{\lambda}$.

To be expanded order-by-order in 1/N.

We can identify the functional determinant with the grand-canonical (fixed chemical potential) free energy:

$$F_{gc}^{\boxplus}(i\theta) = \sum_{i=1}^{N} \left[V \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \frac{1}{\beta} \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 \right) \right] \right].$$

ZETA FUNCTIONS

In the limit $\beta \to \infty$ (zero temperature), we regularize with a zeta function $\zeta(s|\Sigma, m) = \sum_p \left(E(p)^2 + m^2\right)^{-s}$:

The gap equations are (set $A_1 = v$, $A_{>1} = 0$):

$$\frac{\delta}{\delta m} : Vv^2 + \frac{N-1}{2}\zeta(1/2|\Sigma, m) = 0,$$
$$\frac{\delta}{\delta \theta} : -iQ + \frac{2V}{\beta}\theta v^2 = 0,$$
$$\frac{\delta}{\delta v} : 2V\beta\left(m^2 + \frac{\theta^2}{\beta^2}\right)v = 0,$$

For finite Q we need necessarily $v \neq 0$ and then $\theta = im\beta$. So we get

$$m\zeta(1/2|\Sigma, m) = -\frac{Q}{N-1}$$

ORDER N

At leading order in N, the free energy is

$$F(Q) = -\frac{1}{\beta}\left(i\theta Q + N\,\frac{\partial}{\partial s}\frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)}\beta\zeta(s-1/2|\Sigma, m)\Big|_{s=0}\right)$$

Using the gap equations

$$F(Q) = mQ + N\zeta(-1/2|\Sigma, m)$$

For $\Sigma = S^2$ at large Q/N:

$$F(Q) = \frac{N\sqrt{2}}{3}\left(\frac{Q}{N}\right)^{3/2} + \frac{N}{3\sqrt{2}}\left(\frac{Q}{N}\right)^{1/2} - \frac{7N}{180\sqrt{2}}\left(\frac{Q}{N}\right)^{-1/2} + \ldots$$

SMALL Q/N

The zeta function can be expanded in perturbatively in small Q/N.
Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2}\frac{Q}{N} + \frac{16\left(\pi^2 - 12\right)Q^2}{3\pi^4 N^2} + \ldots$$

- Expansion of a closed expression
- It's a convergent series with finite radius of curvature (marks the threshold of the effective field theory (EFT))
- Start with the engineering dimension 1/2
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation

ORDER N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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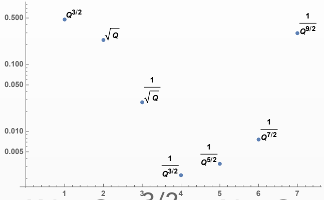

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FINAL RESULT

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{1/2} + \ldots$$

$$- 0.0937\ldots$$

FINAL RESULT

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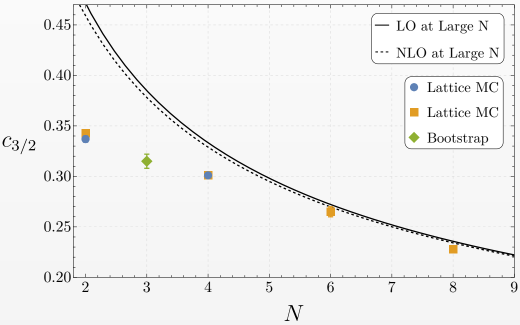
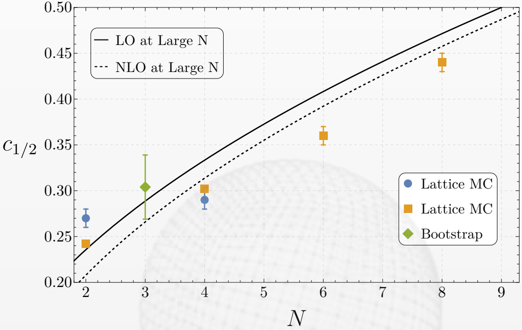
$$- 0.0937\ldots$$





[Dondi & Sberveglieri, arXiv:2409.06781]

FERMIONS AT LARGE N

GROSS–NEVEU AND NAMBU–JONA–LASINIO

Simplest four-Fermi models

$$S_{GN}[\psi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{2N}(\bar{\psi}\psi)^2$$

$$S_{NJL}[\psi, \varphi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{N}\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right]$$

Respectively $O(4N)$ and $U(N) \times U(1)$ symmetry.

Non-trivial CFT in the ultraviolet (UV).

Convenient to introduce collective fields

$$\sigma = \frac{g}{N}\bar{\psi}\psi$$

$$\varphi = \frac{g}{N}\left(\bar{\psi}\psi + \bar{\psi}\gamma^5\psi\right)$$

GNY AND NJL MODELS AT LARGE N

$$S_{GNY}[\psi,\sigma] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \left(\gamma^\mu \partial_\mu + \sigma\right) \psi_i + \frac{1}{2g_Y} \partial_\mu \sigma \partial_\mu \sigma .$$

$$S_{NJL}[\psi,\varphi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \left[\gamma^\mu \partial_\mu + \varphi \left(\frac{1+\gamma_5}{2}\right) + \varphi^* \left(\frac{1-\gamma_5}{2}\right)\right] \psi_i + \frac{1}{g_Y} \partial_\mu \varphi^* \partial_\mu \varphi$$

They flow in the IR to the same CFTs.

We want to fix $U(1)$ charges.

$$\psi_i \to e^{i\alpha} \psi_i, \qquad \sigma \to \sigma$$

$$\psi \to e^{i\alpha\gamma_5} \psi, \qquad \varphi \to e^{-2i\alpha} \varphi,$$

and compute the partition function at fixed charge

$$Z(Q) = \mathrm{Tr}\left[e^{-\beta H} \delta(\hat{Q} - Q)\right].$$

FIX THE CHARGE

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta Q}\, \mathrm{Tr}\left[e^{-\beta H} e^{-i\theta \hat{Q}} \right].$$

At large N the integral over θ becomes a Legendre transform

$$\Delta(Q) = -\frac{1}{\beta} \log\left(Z_{S^2}(Q) \right) = \sup_{i\theta} \left(i\theta Q - S_{\mathrm{eff}}(\theta) \right)$$

The trace is written as a path integral in a flat connection

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\substack{\psi(2\pi\beta) = -\psi(0) \\ \varphi(2\pi\beta) = \varphi(0)}} \mathcal{D}\varphi_i \; e^{-S^{\theta}[\varphi]}$$

EFFECTIVE ACTION: COVARIANT DERIVATIVE

The actions are quadratic in the fermions: Gaussian integral.

For the bosonic fields, we expand as VEV plus fluctuations

$$\sigma = \sigma_0 + \frac{1}{\sqrt{N}}\hat{\sigma}$$

$$\varphi = \Phi_0 + \frac{1}{\sqrt{N}}\hat{\varphi}$$

The leading contribution comes from the VEV:

$$\Omega = S_{eff} = -N\,\mathrm{Tr}\log\left(\gamma^\mu \partial_\mu - i\frac{\theta}{\beta}\gamma_3 + \sigma_0\right)$$

$$\Omega = S_{eff} = -N\,\mathrm{Tr}\log\left(\gamma^\mu \partial_\mu - i\frac{\theta}{\beta}\gamma_3\gamma_5 + \Phi_0\left(\frac{1+\gamma_5}{2}\right) + \Phi_0^*\left(\frac{1-\gamma_5}{2}\right)\right)$$

Gap equation: minimized wrt σ_0 or Φ_0 (gap equation).

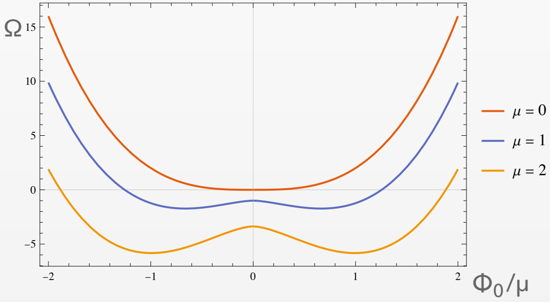
As before, $i\theta/\beta = \mu$ is the chemical potential.

THE NAMBU–JONA–LASINIO MODEL

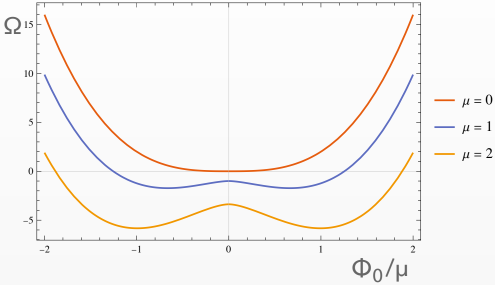
In the $\beta \to \infty$ limit, on the torus $\Sigma = T^2$, the grand potential is

$$\frac{\Omega}{N} = -\int \frac{d^2 p}{(2\pi)^2}\left[\sqrt{(|p|+\mu)^2 + |\Phi_0|^2} + \sqrt{(|p|-\mu)^2 + |\Phi_0|^2}\right]$$

$$= -\frac{1}{6\pi}\left[3|\Phi_0|^2 \mu \operatorname{arctanh}\frac{\mu}{\sqrt{|\Phi_0|^2+\mu^2}} + (\mu^2 - 2|\Phi_0|^2)\sqrt{|\Phi_0|^2 + \mu^2}\right]$$



THE NAMBU–JONA–LASINIO MODEL



There is a minimum (gap equation) for every value of μ μ

$$\Phi_0 = \mu\sqrt{\kappa_0^2 - 1} = 0.6627\cdots\mu$$

Same physics: fixing the charge induces a spontaneous symmetry breaking.
The field φ is the order parameter for the **superfluid phase transition**.

There is only **one difference**.
For bosons, zero modes fix the gap as function of the chemical potential m = μ.
Fermions don't have zero modes and the gap is fixed by a one-loop term.

COOPER PAIRS

The scalar φ is a composite field

$$\varphi = \bar{\psi}\psi + \bar{\psi}\gamma^5\psi$$

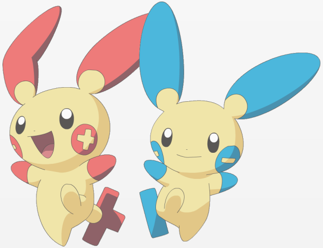
More transparent after a Pauli-Gürsey transformation:

$$\psi \mapsto \frac{1}{2}\left[(1-\gamma_5)\psi - (1+\gamma_5)C\bar{\psi}^T\right], \qquad \bar{\psi} \mapsto \frac{1}{2}\left[\bar{\psi}(1+\gamma_5) - \psi^T C(1-\gamma_5)\right].$$

because then we identify φ with a Cooper pair

$$\varphi = \psi^t C\psi$$

With attractive interactions, fermion form pairs and condense.



CONFORMAL DIMENSIONS @NJL

Repeat the computation on S^2.

Infinite sum to regularize: zeta function for the Dirac operator.

The **action at the saddle** is

$$S(\mu, \Phi_0) = -\beta N \zeta(-1/2 | \not{\partial}, \mu, \Phi_0),$$

$$\zeta(s|\not{\partial}, \mu, \Phi_0) = 2r^{2s} \sum_{\ell=1}^{\infty} \ell \left[\left((\ell + r\mu)^2 + (r\Phi_0)^2 \right)^{-s} + \left((\ell - r\mu)^2 + (r\Phi_0)^2 \right)^{-s} \right].$$

1. Minimize with respect to Φ_0 for the gap equation.
2. Legendre transform from the chemical potential μ to the charge $q = Q/(2N)$.

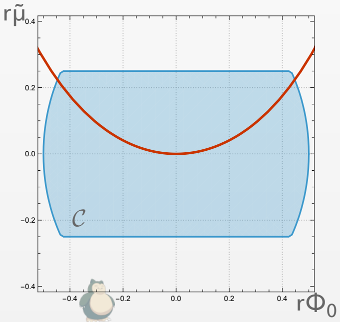
SMALL CHARGE

$$\frac{S(\tilde{\mu},\Phi_0)}{\beta N} = -\frac{\pi^2 r^2 \tilde{\mu}}{8}(\Phi_0^2+\tilde{\mu}^2) - \frac{3\pi^2 r^3 \tilde{\mu}^2}{8}(\Phi_0^2+\tilde{\mu}^2) + \frac{\pi^2 r^3}{16}(\Phi_0^2+\tilde{\mu}^2)^2 + \dots$$

And the gap equation is

$$r\tilde{\mu} = (r\Phi_0)^2 + \frac{\pi^2-8}{4}(r\Phi_0)^4 + \frac{192-8\pi^2-\pi^4}{24}(r\Phi_0)^6 + \dots,$$

Finite region of convergence



SMALL CHARGE

1. invert the gap equation and write the gap Φ_0 as function of the chemical potential μ,
2. Legendre transform from μ to $q = Q/(2N)$

Conformal dimensions in the small q limit

$$\frac{\Delta}{2N} = \frac{1}{2}q + \frac{2}{\pi^2}q^2 + \frac{8\left(\pi^2 - 12\right)}{3\pi^4}q^3 + \frac{2\left(\pi^2 - 48\right)\left(3\pi^2 - 32\right)}{3\pi^6}q^4 + \ldots$$

Consistent with $\Delta(Q) \approx Q/2$.
Once more, it is the resummation of an infinite number of diagrams.

The series can be computed at any given order, and it has a finite radius of convergence $|q| \leq 0.35(3)$.

LARGE CHARGE @NJL

Use a Mellin representation (heat kernel)

$$\zeta(s|\not{\partial}, \mu, \Phi_0) = \frac{2}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\Phi_0^2 t} \sum_{\ell=1}^{\infty} \ell \left(e^{-(\ell/r+\mu)^2 t} + e^{-(\ell/r-\mu)^2 t} \right) .$$

Action at the saddle

$$\frac{S(\mu, \Phi_0)}{\beta N} = \left(\frac{\mu^2 - 2\Phi_0^2}{3} \sqrt{\mu^2 + \Phi_0^2} + \mu \Phi_0^2 \operatorname{arcsinh}\left(\frac{\mu}{\Phi_0}\right) \right) r^2 + \frac{1}{6} \sqrt{\mu^2 + \Phi_0^2}$$
$$+ \frac{\Phi_0^2}{120 \left(\mu^2 + \Phi_0^2\right)^{3/2} r^2} + \frac{\Phi_0^4 - 4\mu^2 \Phi_0^2}{1008 \left(\mu^2 + \Phi_0^2\right)^{7/2} r^4} + \dots$$

LARGE CHARGE

@NJL

1. invert the gap equation and write the gap Φ_0 as function of the chemical potential μ,
2. Legendre transform from μ to Q/N

ORDER N

@NJL

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{1/2} - \frac{11-6\kappa_0^2}{360\kappa_0^2}\left(\frac{Q}{2N\kappa_0}\right)^{-1/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

ORDER N

@NJL

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{1/2} - \frac{11 - 6\kappa_0^2}{360\kappa_0^2}\left(\frac{Q}{2N\kappa_0}\right)^{-1/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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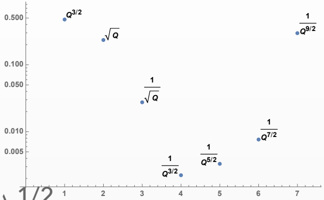
ORDER N



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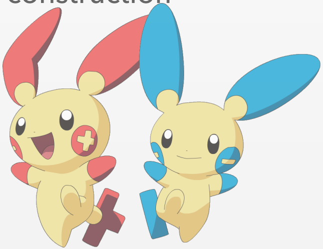
WHAT HAS HAPPENED?

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge results in **spontaneous symmetry breaking**
- The order parameter is a composite field (**Cooper pair**)
- We compute the conformal dimension of the lowest operator of charge Q
- We find a result in **total agreement** with the general EFT construction



THE GROSS–NEVEU–YUKAWA MODEL

Now for the Gross-Neveu-Yukawa model.

Integrating out the Matsubara frequencies we find

$$\frac{\Omega}{N} = -2\int \frac{d^2 p}{(2\pi)^2}\left\{\omega_p + \frac{1}{\beta}\log\left(1 + e^{-\beta(\omega_p+\mu)}\right) + (\mu \leftrightarrow -\mu)\right\},$$

with $\omega_p^2 = p^2 + \sigma_0^2$. The gap equation is

$$\sigma_0 - \frac{1}{\beta}\log\Big((1 + e^{\beta(\sigma_0+\mu)})(1 + e^{\beta(\sigma_0-\mu)})\Big) = 0$$

and admits only the solution $\sigma_0 = 0$.

In other words, in the large-N limit the symmetry is never broken.

This is different from the superfluid EFT behavior.

THE GROSS–NEVEU–YUKAWA MODEL

We can repeat the computation on the sphere. The grand potential is

$$\frac{\Omega}{N} = -\frac{1}{2\pi r_0^2}\left[\sum_{\omega_j > \mu}(2j+1)\omega_j + \mu \sum_{\omega_j < \mu}(2j+1)\right]$$

the solution to the gap equation and the Legendre transform are

$$\frac{Q}{N} = \frac{1}{2\pi r_0^2}\lfloor \mu r_0 \rfloor(\lfloor \mu r_0 \rfloor + 1), \qquad \frac{E}{N} = \frac{1}{6\pi r_0^3}\lfloor \mu r_0 \rfloor(\lfloor \mu r_0 \rfloor + 1)(2\lfloor \mu r_0 \rfloor + 1).$$

This is the physics of a Fermi sphere.

However, the behavior of the conformal dimension is still the same

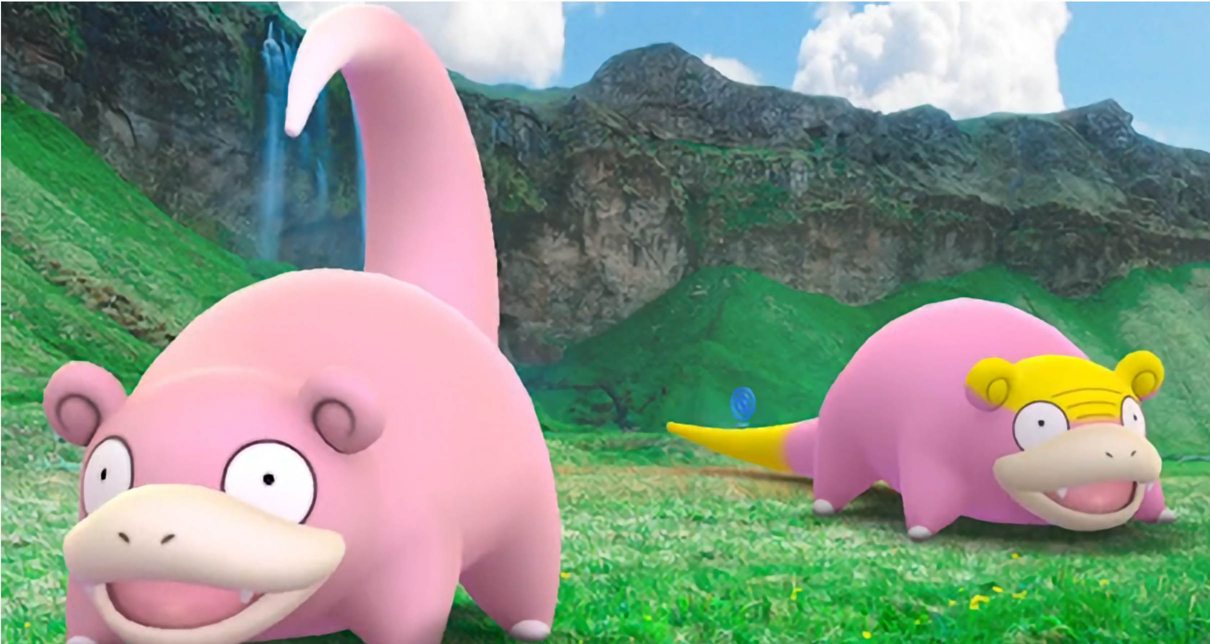
$$\Delta = \frac{2}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{1}{12}\left(\frac{Q}{2N}\right)^{1/2} - \frac{1}{192}\left(\frac{Q}{2N}\right)^{-1/2} + \dots$$

WHAT HAS HAPPENED?

@GNY

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge **does not result** in spontaneous symmetry breaking
- The physics is the one of a **Fermi sphere**
- The dimension of the lowest operator still obeys the same law
- The Fermi sphere is unstable if there is an attractive channel: there must be a finite-N transition and an **exponentially-suppressed gap**

NRCFT



AND NOW FOR SOMETHING COMPLETELY DIFFERENT: THE SCHRÖDINGER GROUP

The Schrödinger group describes the symmetries of the **free-particle Schödinger equation**

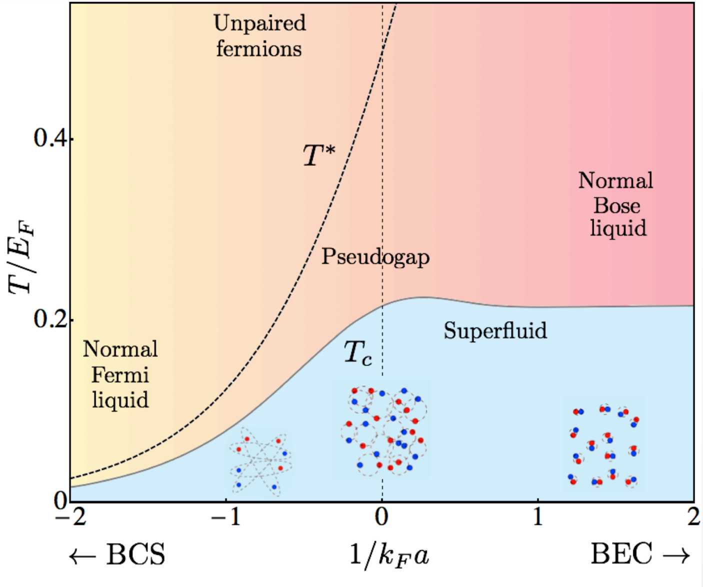
$$i\frac{\partial}{\partial t}\psi = -\frac{\triangle}{2}\psi$$

Its algebra contains the **Galileian algebra**, plus a **central extension** and **two more generators** that, together with time translations form an $SL(2,\mathbb{R})$ algebra.

Contraction of the conformal group, just like the Galileian group is a contraction of the Poincaré group.

It is the symmetry of a **non-relativistic conformal field theory**.

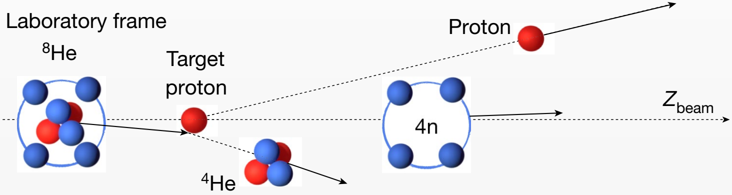
ATOMS AT CRITICALITY

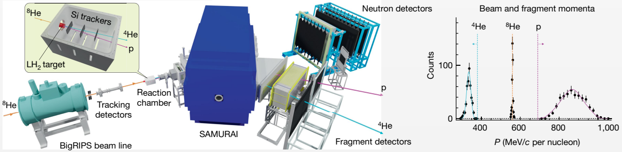


UNNUCLEAR PHYSICS

Neutrons have a **large negative scattering length $a \approx -19$ fm** compared to the effective range of the interaction $r_0 \approx 2.8$fm.

There is a range of energies in which a system of neutrons behaves as a **composite object**, controlled by a NRCFT.



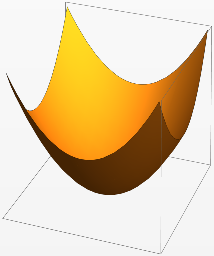


STATE-OPERATOR CORRESPONDENCE

State operator correspondence in CFT: operators inserted in $\mathbb{R}^d$ are mapped to states on $\mathbb{R} \times S^{d-1}$. The conformal dimension maps to energy.

In NRCFT there is a similar story.

Operators in $\mathbb{R}^d$ → states on $\mathbb{R}^d$ with a background harmonic potential.

$A_0 =$ 

To compute $\Delta(Q)$ we add a background field and compute the energy.
It is the same problem!

THE SCHRÖDINGER ACTION AND THE LARGE-CHARGE EFT

The simplest **microscopic model** is

$$L_{micro} = \begin{pmatrix} \psi_{\uparrow a} \\ \bar{\psi}_{\downarrow a} \end{pmatrix}^t \begin{pmatrix} -\partial_\tau + \frac{1}{2}\triangle + \mu - A_0(\mathbf{r}) & \sigma(\tau, \mathbf{r}) \\ \sigma(\tau, \mathbf{r})^* & -\partial_\tau - \frac{1}{2}\triangle - \mu + A_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} \psi_{\uparrow a} \\ \bar{\psi}_{\downarrow a} \end{pmatrix} - \frac{N}{2u_0}\sigma^* \sigma$$

The **large-charge EFT** at the unitary point is

$$L_{EFT} = c_0 U^{5/2} + c_1 \frac{(\nabla U)^2}{U^{1/2}} + c_2 U^{1/2} \left[(\triangle \chi)^2 - 3(\nabla \otimes \nabla \chi)^2 \right]$$

where

$$U = \partial_t \chi - A_0 - \frac{1}{2}\partial_i \chi \partial_i \chi$$

THE MOYAL EXPANSION

It's again the same problem

$$\mathrm{Tr}\log\left(G^{-1}\right) = \mathrm{Tr}\log\begin{pmatrix} -\partial_\tau + \frac{1}{2}\triangle + \mu - A_0(\mathbf{r}) & \sigma(\tau,\mathbf{r}) \\ \sigma(\tau,\mathbf{r})^* & -\partial_\tau - \frac{1}{2}\triangle - \mu + A_0(\mathbf{r}) \end{pmatrix}$$

There still is time-translation invariance, so we can trace over τ:

$$G^{-1} = -\partial_\tau + B(\mathbf{r}) \Rightarrow \mathrm{Tr}\log G^{-1} = \frac{\beta}{2}\mathrm{Tr}\begin{vmatrix} \frac{1}{2}\triangle + \mu - A_0(\mathbf{r}) & \sigma(\tau,\mathbf{r}) \\ \sigma(\tau,\mathbf{r})^* & -\frac{1}{2}\triangle - \mu + A_0(\mathbf{r}) \end{vmatrix}$$

but now the operator contains $\triangle$, $A_0(\mathbf{r})$ and $\sigma(\mathbf{r})$ that do not commute.

Formally we are still good

$$\mathrm{Tr}\left(B^{-s}\right) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t} t^s \,\mathrm{Tr}\left(e^{-tB}\right) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t} t^s \,\mathrm{Tr}(K(t))$$

but how to compute this heat kernel?

MOYAL EXPANSION

The **heat kernel** solves the equation

$$\left(\partial_t + B(\mathbf{r}_1)\right) K(t; \mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1, \mathbf{r}_2)$$

These are just the matrix elements of $K(t)$ in the position basis.

Think of them as **bilocal operators** with their w_∞ algebra

$$\int d\mathbf{r}_2 \, A(\mathbf{r}_1, \mathbf{r}_2) B(\mathbf{r}_2, \mathbf{r}_3) = C(\mathbf{r}_1, \mathbf{r}_3).$$

MOYAL EXPANSION

$$\begin{cases} r = r_1 - r_2 \\ R = 2\left(r_1 + r_2\right) \end{cases}$$

and Fourier transform over the relative position r

$$A(p, R) = \int dr\, e^{ipr/\hbar} A\left(r - \frac{R}{2}, r + \frac{R}{2}\right)$$

The convolution becomes a **star product**

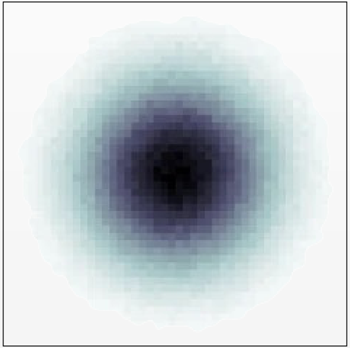
$$A \star B = A \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_R \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_R\right)\right] B$$

and if ħ is small we have an algorithmic way of computing the heat kernel (and solve our original problem)

$$A \star B = AB + \frac{i\hbar}{2}[A, B] + \ldots$$

WHO IS ħ?

The harmonic trap **breaks translation invariance**. The ground state configuration is not homogeneous, but it's a **cloud**. The density at the edge goes to zero



This introduces a **scale** $R_{cl} = \sqrt{\mu}/\omega$.

WHO IS ħ?

Use R_{cl} to rescale the coordinates and identify

$$\hbar = \frac{1}{\sqrt{\mu}}\frac{1}{R_{cl}} = \frac{\omega}{\mu}$$

The large chemical potential controls the derivative expansion.

It is the ħ that controls the semiclassical expansion. **On the nose**.

Now just turn the crank.

There is SSB and everything is written in terms of an hypergeometric function

$$\frac{I_{0,0}(y)}{y^{5/2}} = \Gamma(-\tfrac{5}{4})\Gamma(\tfrac{3}{4})\,{}_2F_1(-\tfrac{5}{4},-\tfrac{1}{4},\tfrac{1}{2};-\tfrac{1}{y^2}) + \frac{2}{y}\Gamma(-\tfrac{3}{4})\Gamma(\tfrac{5}{4})\,{}_2F_1(-\tfrac{3}{4},\tfrac{1}{4},\tfrac{3}{2};-\tfrac{1}{y^2}).$$

$$\frac{\Delta}{N} = 0.8313\left(\frac{Q}{N}\right)^{4/3} + 0.2631\left(\frac{Q}{N}\right)^{2/3} + \dots$$

WHAT HAS HAPPENED?

- We can use the same construction for **non-relativistic systems**
- Direct **experimental** applications
- Technically everything becomes more complicated because of **broken translation invariance**
- We can use a semiclassical expansion and **large charge is literally** $1/\hbar$
- The problem can be solved at any given order in $1/Q$

RESURGENCE AND THE LARGE CHARGE

RESULTS FROM LARGE N

O(2N) at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $q = Q/(2N)$ fixed.

$$\begin{cases} F_{\Sigma}(Q) = \mu Q + N\zeta(-\frac{1}{2}|\Sigma, \mu), \\ \mu\zeta(\frac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}. \end{cases}$$

RESULTS FROM LARGE N

O(2N) at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $q = Q/(2N)$ fixed.

The free energy per DOF $f(q) = F/(2N)$ is

$$f(q) = \sup_{\mu}(\mu q - \omega(\mu)), \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

RESULTS FROM LARGE N

O(2N) at criticality in $1+2$ dimensions on $\mathbb{R}\times\Sigma$. Double-scaling limit $N\to\infty$, $Q\to\infty$ with $q = Q/(2N)$ fixed.

The free energy per DOF $f(q) = F/(2N)$ is

$$f(q) = \sup_{\mu}(\mu q - \omega(\mu)), \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma,\mu),$$

$\zeta(s|\Sigma,\mu)$ is the zeta function for the operator $-\triangle+\mu^2$. In Mellin representation

$$\zeta(s|\Sigma,\mu) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t}\,\mathrm{Tr}\left(e^{\triangle t}\right).$$

RESULTS FROM LARGE N

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$$\zeta(s|\Sigma, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t} \operatorname{Tr}\left(e^{\triangle t}\right).$$

Large q is large μ and is small t. The classical Seeley-de Witt problem:

$$\operatorname{Tr}\left(e^{\triangle t}\right) \sim \frac{V}{4\pi t}\left(1 + \frac{R}{12} t + \dots\right).$$

THE TORUS

As a warm-up: $\Sigma = T^2$.

$$\mathrm{spec}(\triangle) = \left\{ -\frac{4\pi^2}{L^2}\left(k_1^2 + k_2^2\right) \mid k_1, k_2 \in \mathbb{Z} \right\}.$$

It follows that the heat kernel trace is the square of a theta function:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}(k_1^2 + k_2^2)t} = \left[\theta_3\left(0, e^{-\frac{4\pi^2 t}{L^2}}\right)\right]^2.$$

We are interested in the small-t limit: we Poisson-resum the series:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1 + \sum_{k \in \mathbb{Z}} e^{-\frac{k^2 L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{\|k\|^2 L^2}{4t}}\right)$$

THE TORUS

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|T^2,\mu) = \frac{L^2\mu^3}{12\pi}\left(1 + \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\mu L}}{\|\mathbf{k}\|^2\mu^2 L^2}\left(1 + \frac{1}{\|\mathbf{k}\|\mu L}\right)\right).$$

Free energy

$$f(q) = \sup_{\mu}(\mu q - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L} q^{3/2}\left(1 - \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\sqrt{4\pi q}}}{8\|\mathbf{k}\|^2\pi q} + \dots\right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/q$ and $e^{-\sqrt{4\pi q}}$.
- non-perturbative effects more important than the "usual" instantons $\mathcal{O}(e^{-q})$

THE SPHERE

On the two sphere $\mathrm{spec}(\triangle) = \{-\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell + 1$.

Again, we use Poisson resummation

$$\mathrm{Tr}\left(e^{\triangle t}\right) e^{-t/4} = \sum_{\ell \geq 0} (2\ell + 1) e^{-(\ell+1/2)^2 t} \sim \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(1 - 2^{1-2n})}{n!} B_{2n} t^n$$

The series is asymptotic: the Seeley-de Witt coefficients diverge like n!:

$$a_n = \frac{(-1)^{n+1}(1 - 2^{1-2n})}{n!} B_{2n} \sim \frac{2n^{1/2}}{\pi^{5/2+2n}} n!.$$

this divergence is reflected in the existence of non-perturbative corrections.

BOREL RESUMMATION

BOREL TRANSFORM

We need to make sense of the divergent series and the imaginary terms.

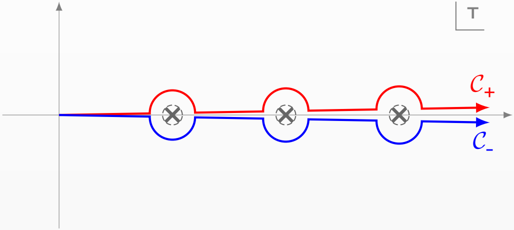
$$H(t) = \sum_{n\geq 0} a_n t^n$$

$$\hat{H}(\tau) = \sum_{n\geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

LATERAL TRANSFORM

If there are poles on the real positive axis there is an ambiguity



$$s_{\pm}(H)(t) = \int_{\mathcal{C}_{\pm}} w^b e^{-w} \hat{H}(tw^{\beta}) \frac{dw}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.

MORE INGREDIENTS

WORLDLINE INTERPRETATION

We need a **non-perturbative interpretation** of these exponential terms.

The heat kernel is the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \triangle$, i.e. a **free quantum particle moving on $\mathbb{R} \times \Sigma$**.

We can write the partition function as a **path integral**

$$\mathrm{Tr}\left(e^{(\partial_0^2 + \triangle)t}\right) = \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t}\int_0^1 d\tau\, g_{\mu\nu}\dot{X}^\mu(\tau)\dot{X}^\nu(\tau)$$

A TRANSSERIES FROM GEODESICS

For $t \to 0$ the path integral localizes on a sum over all the closed geodesics γ.
For each geodesic a perturbative series in t, weighted by $e^{-\ell(\gamma)^2/(4t)}$

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

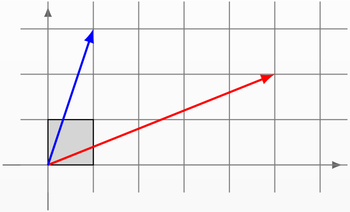
$$= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma\, \in \text{ closed geodesics}} e^{-\frac{\ell(\gamma)^2}{4t}}\, t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,$$

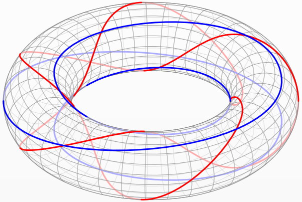
the b_γ depend on the geometry.

This is precisely the same structure predicted by resurgence.
Now we have a geometric interpretation.

THE TORUS

In the case of the torus, closed geodesics are labelled by two integers (k_1, k_2)





The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h \, e^{-\frac{1}{4t}\int_0^1 d\tau (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t}\partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$

THE TORUS

Now we can write the result of the path integral

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{cl}} \int_{h(1)=h(0)=0} e^{-S[X_{cl}]-S[h]}$$

$$= \mathcal{N} L^2 \sum_{k\in\mathbb{Z}^2} e^{-\frac{L^2(k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h\, e^{-S[h]},$$

$$= \frac{L^2}{4\pi t}\left[1 + \sum_{k\in\mathbb{Z}^2} e^{-\frac{L^2\|k\|^2}{4t}}\right]$$

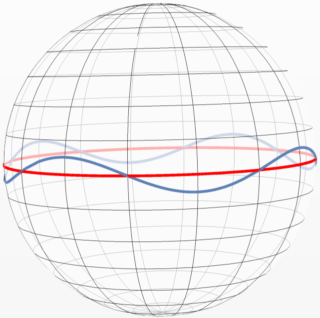
This is exactly what we had found before just by looking at the spectrum.
The non-perturbative effects come from closed geodesics.

THE SPHERE

Closed geodesics on the sphere go around the equator k times

THE SPHERE

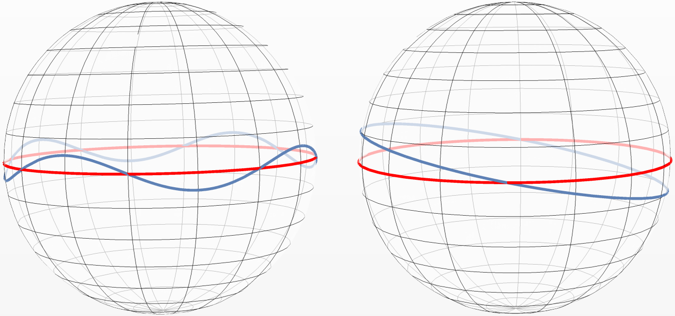
Closed geodesics on the sphere go around the equator k times



We need to sum over the fluctuations h_φ and h_θ

THE SPHERE

Closed geodesics on the sphere go around the equator k times



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There is a zero mode because we can rotate the equator

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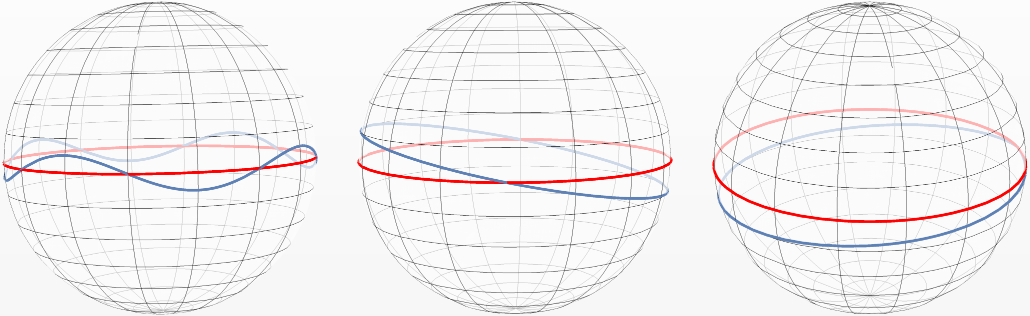

We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator

And an instability because we can slide off

BACK TO RESURGENCE

Putting it **all together**, the non-trivial geodesics give

$$\pm 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\in\mathbb{Z}}|k|e^{-\frac{k^2\pi^2}{t}}$$

(the i comes from the instability).

The one-loop result **perfectly cancels** the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr}\left(e^{(\triangle-\frac{1}{4})t}\right) = s_{\pm}(H)(t) \mp 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\geq 1}(-1)^k k e^{-\frac{k^2\pi^2}{t}} = \mathrm{Re}[s_{\pm}(H)(t)]$$

BACK TO RESURGENCE

We can write the **exact expression** for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \mathrm{Re}\left[\frac{2rm^2}{\pi}\int_0^\infty dy\,\frac{K_2(2mry)}{y\sin(y)}\right] = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

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As a numerical test, we can compare with the convergent small-charge expansion ($Q \approx 1.2 \cdot N$)

$$r\omega(mr = 0.4)\Big|_{\text{small charge}} = 0.012\,777\,296\,63\ldots$$

$$r\omega(mr = 0.4)\Big|_{\text{resurgence}} = 0.012\,777\,297\,69\ldots$$

OPTIMAL TRUNCATION

LESSONS FROM LARGE N

Let's go back to the EFT.
The effective action is identified with the asymptotic expansion: the **grand potential** is the value of the **action at the minimum $\chi = \mu t$**:

$$\omega(\mu) = L_{EFT}\Big|_{\chi=\mu t}$$

where

$$L_{EFT} = \omega_0 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{3/2} + \omega_1 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{1/2} + \dots,$$

In general the **coefficients are unknown**

BUT

Now we have a **geometric understanding** of the non-perturbative effects

LESSONS FROM LARGE N

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is
   a particle of mass μ going around the equator.

A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n\geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1\, Q^{b_1}\, e^{-3\pi\kappa f_0^{(0)} \sqrt{Q}} \sum_{n\geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \dots$$

(we used $\mu = 3 f_0^{(0)} \sqrt{Q}/2 + \dots$.)

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LESSONS FROM LARGE N

- The **controlling parameter** for the non-perturbative effects $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ is fixed by the **leading term** in the 1/Q expansion.
- The non-perturbative coefficient $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ fixes the **large-n behavior** of the perturbative series $f_n^{(0)}$.

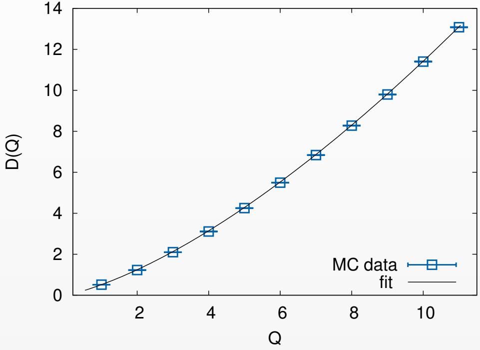
$$f_n^{(0)} \sim (2n)!(3\pi\kappa f_0^{(0)})^{-n}$$

We don't know enough for a Borel resummation, but we can estimate an optimal trucation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi\kappa f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$

CAN WE UNDERSTAND THE LATTICE RESULTS NOW?



In O(2), $f_0^{(0)} \approx 0.301(3)$

so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\pi\sqrt{Q}}\right)$.

Lattice: Best fit with N = 3 terms.
At Q = 1 the error is $\approx 6 \times 10^{-2}$; at Q = 11 the error is $\approx 5 \times 10^{-5}$.

Resurgence: $\sqrt{10} \approx 3.16$
$e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$.

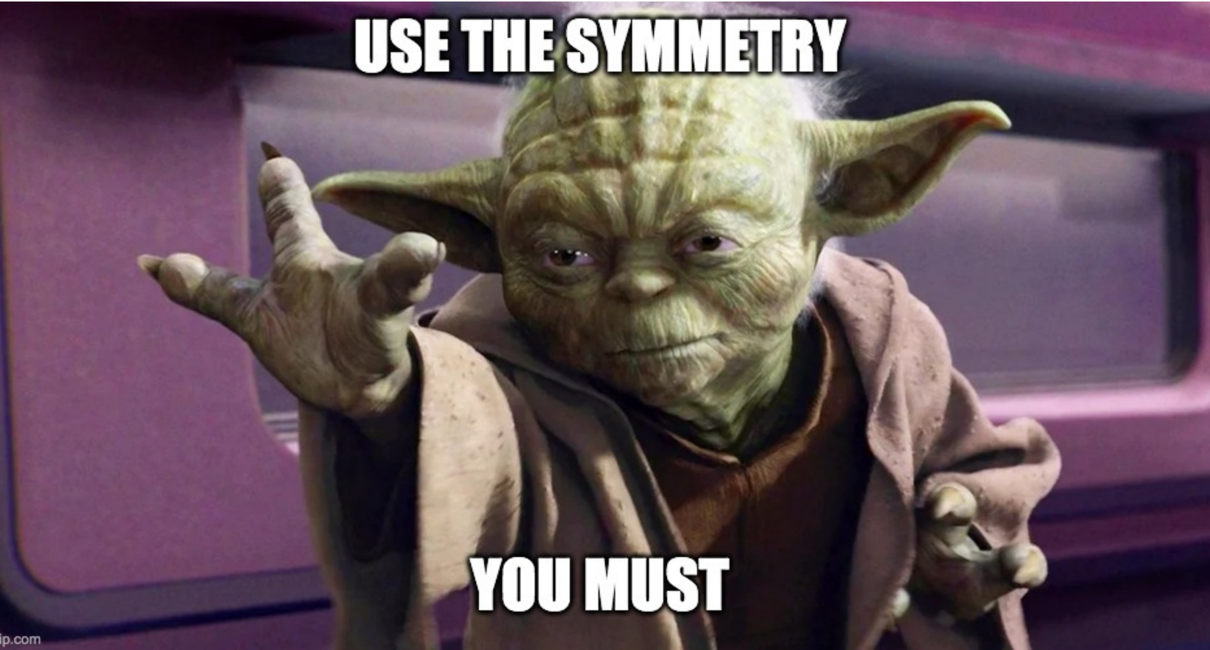
WHAT HAS HAPPENED?

- The large-charge expansion of the Wilson-Fisher point is **asymptotic**
- In the **double-scaling** limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion
- We can **Borel**-resum the expansion
- We have a **geometric interpretation for the non-perturbative effects**
- We can use this geometric interpretation also in the **finite-N** case
- We obtain an **optimal truncation** and estimate of the error
- The results are **consistent with lattice simulations**

CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Precise and **testable predictions**.
- Qual(nt)itative control of the **non-pertubative** effects.
- **CFT constraints**: perturbative/non-perturbative **interplay**.
- Remarkable agreement with **lattice**.

AN EFT FOR A CFT



THE O(2) MODEL

The simplest example is the WF point of the O(2) model in three dimensions.

- Non-trivial fixed point of the φ^4 action

$$L_{UV} = \partial_\mu \varphi^* \partial_\mu \varphi - u(\varphi^* \varphi)^2$$

- Strongly coupled
- In nature: ^{4}He.
- Simplest example of spontaneous symmetry breaking.
- **Not accessible** in perturbation theory. **Not accessible** in $4 - \varepsilon$. **Not accessible** in large N.
- Lattice. Bootstrap.

CHARGE FIXING

We consider a **subsector of fixed charge** Q.

Generically, the classical solution at fixed charge **breaks spontaneously** $U(1) \rightarrow \emptyset$.

We have one **Goldstone boson** χ.

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

AN ACTION FOR χ

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$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

We want to describe a CFT: we can **dress with a dilaton**

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken U(1), the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

LINEAR SIGMA MODEL

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.

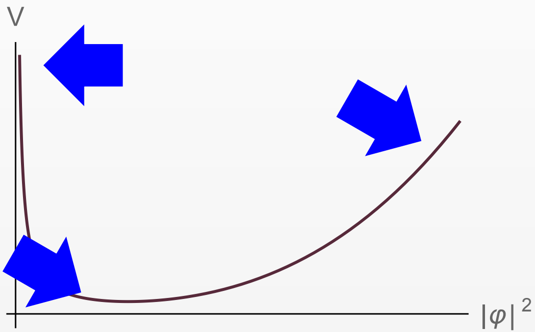
Scale invariance is manifest.

The field φ is some complicated function of the original φ.

CENTRIFUGAL BARRIER

The O(2) symmetry acts as a shift on χ.

Fixing the charge is the same as adding a **centrifugal term** $\propto \frac{1}{|\varphi|^2}$.



GROUND STATE

We can find a fixed-charge solution of the type

$$\chi(t, x) = \mu t \qquad \sigma(t, x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \ldots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2}/\sqrt{V}Q^{3/2} + c_{1/2}R\sqrt{V}Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

FLUCTUATIONS

The fluctuations over this ground state are described by two modes.

- A universal "**conformal Goldstone**". It comes from the breaking of the U(1).

$$\omega = \frac{1}{\sqrt{2}} p$$

- The **massive dilaton**. It controls the magnitude of the quantum fluctuations. **All quantum effects are controled by 1/Q.**

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

NON-LINEAR SIGMA MODEL

Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R(\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.
All other terms are suppressed by powers of 1/Q.

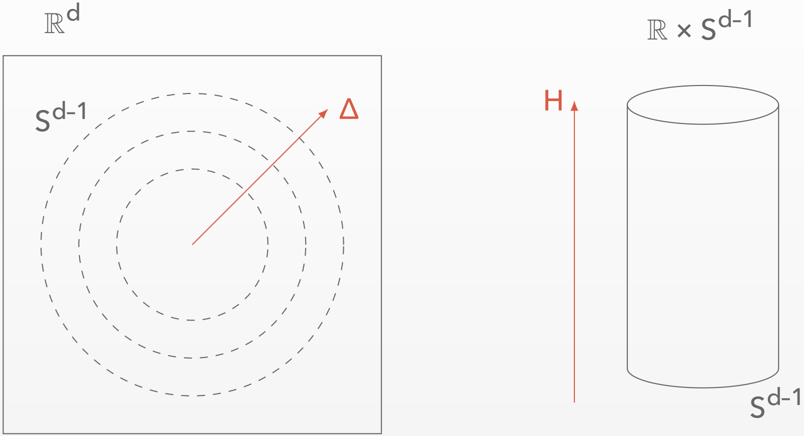
In 3 + 1 NRCFT the analogous story in a background potential A_0 leads to

$$L[\chi] = c_0 U^{5/2} + c_1 U^{-1/2} \partial_i U \partial_i U + c_2 U^{1/2}((\partial_i \partial_i \chi)^2 - 9 \partial_i \partial_i A_0) + \dots \qquad (1)$$

where $U = \partial_t \chi - A_0 \chi - \partial_i \chi \partial_i \chi / 2$.

STATE-OPERATOR CORRESPONDENCE

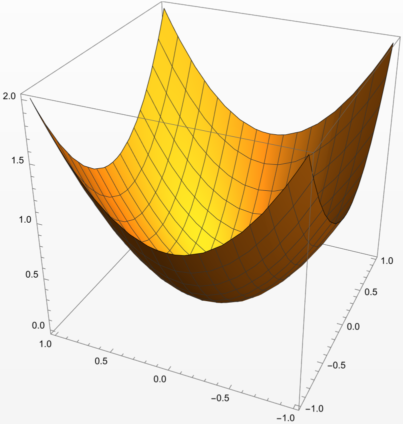
The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.

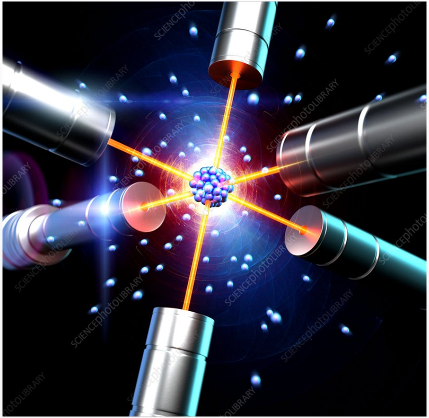


Protected by conformal invariance: a well-defined quantity.

NRCFT STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in a harmonic trap.





Protected by conformal invariance: a well-defined quantity.

CONFORMAL DIMENSIONS

We know the energy of the ground state.

The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}}\zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the **conformal dimension of the lowest operator of charge Q** in the O(2) model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094\ldots + \mathcal{O}\left(Q^{-1/2}\right)$$

In 3 + 1 NRCFT we find

$$\Delta_Q = c_{4/3} Q^{4/3} + c_{2/3} Q^{2/3} + b_{5/9} Q^{5/9} + b_{1/3} Q^{1/3} + b_{1/9} Q^{1/9} - \frac{1}{3\sqrt{3}} \log(Q) + c_0$$

WHAT HAPPENED?

We started from a CFT.
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple EFT**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

ORDER N°

The order N^0 terms are

$$S^{\theta}[\hat{\sigma},\hat{\lambda}] = \int dt\, d\Sigma \left((D_\mu \hat{\sigma})^* (D^\mu \hat{\sigma}) + (\mu^2 + \hat{\lambda})\hat{\sigma}^* \hat{\sigma} + \frac{\hat{\lambda} v(\hat{\sigma} + \hat{\sigma}^*)}{(N-1)^{1/2}} \right)$$
$$+ \frac{1}{2} \int dx_1\, dx_2\, \hat{\lambda}(x_1)\hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.

At low energies we can approximate the non-local term as

$$\int dt\, d\Sigma\, \hat{\lambda}(x)^2 \zeta(2|\theta,\Sigma,\mu) \approx \frac{V}{2\mu} \int dt\, d\Sigma\, \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

ORDER N°

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4\mu^2) & \mu\omega \\ -\mu\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \ldots = 0 \qquad \omega^2 + 8\mu^2 + \frac{3}{2}p^2 + \ldots = 0$$

This is the conformal Goldstone that we have seen in the EFT.

Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\ldots$$

This is **universal**. Does not depend on N or Q.

HIGHER ORDERS

There are infinite non-local terms

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx_1 \dots dx_n \, \hat{\lambda}(x_1)\dots\hat{\lambda}(x_n) P(x_1, \dots, x_n)$$

At low energy they are approximated by

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx \, \hat{\lambda}(x)^n C_n$$

HIGHER ORDERS

There is only one scale, the charge density $\rho = Q/V$. We must have

$$C_n = \rho^{3/2-n} c_n$$

So

$$S_{nl} = Q^{3/2} \sum_{n=3}^{\infty} \frac{c_n}{n(N-1)^{n/2-1}} \int dx\, \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by $1/N$.

back to the talk