

The Resurgence of the Large Charge expansion

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[arXiv:2102.12488](#), and [arXiv:2505.21631](#)



WHAT HAPPENED?

We started from a conformal field theory (CFT).

There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.

We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

» would you like to know more?



TODAY'S TALK

Justify and prove all these claims from first principles



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Justify and prove all these claims from first principles

Use resurgence for the large-charge EFT



TODAY'S TALK

Justify and prove all these claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large- N

Use resurgence for the large-charge EFT



TODAY'S TALK

Justify and prove all these claims from first principles

Use resurgence for the large-charge EFT

- Borel resum the double-scaling $Q \rightarrow \infty, N \rightarrow \infty$ limit
- geometric interpretation of non-perturbative effects
- general structure of the corrections in the EFT

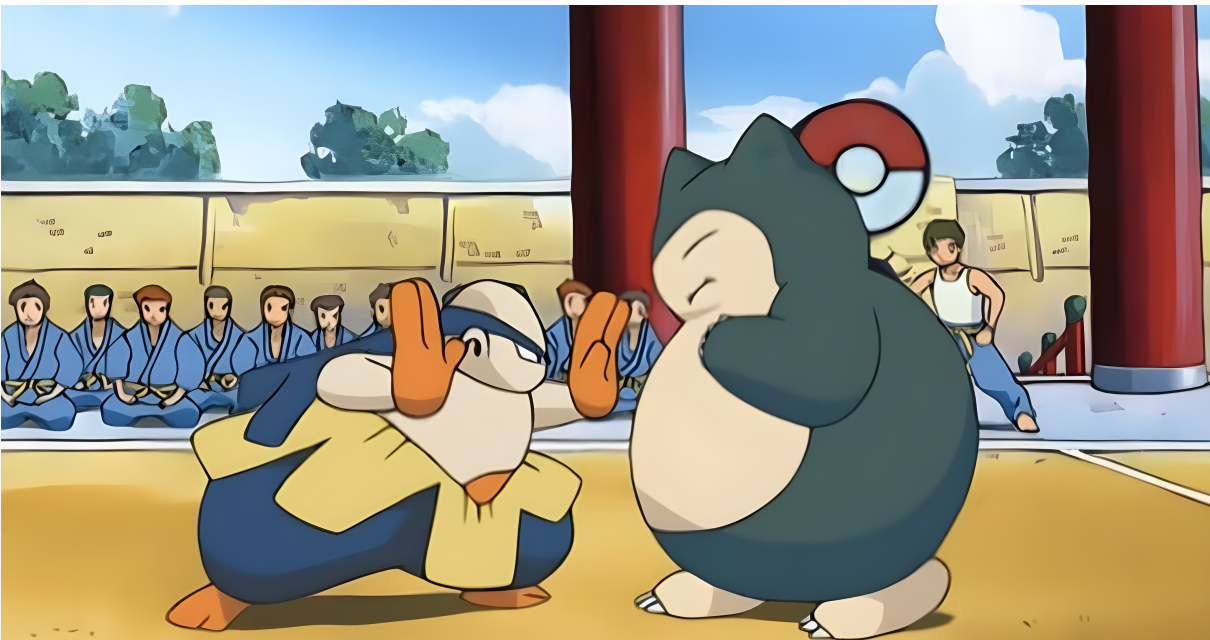


P A R E N T A L

A D V I S O R Y

E X P L I C I T C O N T E N T

LARGE N VS. LARGE CHARGE



THE MODEL

φ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^N \int dt d\Sigma \left[g^{\mu\nu} (\partial_\mu \varphi_i)^* (\partial_\nu \varphi_i) + r \varphi_i^* \varphi_i + \frac{u}{2} (\varphi_i^* \varphi_i)^2 \right]$$

It flows to the WF in the IR limit $u \rightarrow \infty$ when r is fine-tuned.

We compute the partition function at fixed charge

$$Z(Q_1, \dots, Q_N) = \text{Tr} \left[e^{-\beta H} \prod_{i=1}^N \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int d\Sigma j_i^0 = i \int d\Sigma [\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i].$$



FIX THE CHARGE

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^N \frac{d\theta_i}{2\pi} \prod_{i=1}^N e^{i\theta_i Q_i} \text{Tr} \left[e^{-\beta H} \prod_{i=1}^N e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$\begin{aligned} Z_{\Sigma}(Q) &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i e^{-S[\varphi]} \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i e^{-S^{\theta}[\varphi]} \end{aligned}$$



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EFFECTIVE ACTION: COVARIANT DERIVATIVE

$$S^\theta[\varphi] = \sum_{i=1}^N \int dt d\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u(\varphi_i^* \varphi_i)^2 \right)$$

$$\begin{cases} D_0 \varphi = \partial_0 \varphi + i \frac{\theta}{\beta} \varphi \\ D_i \varphi = \partial_i \varphi \end{cases}$$

Stratonovich transformation

$$S_Q = \sum_{i=1}^N \left[-i\theta_i Q_i + \int dt d\Sigma \left[(D_\mu^i \varphi_i)^* (D_\mu^i \varphi_i) + (r + \lambda) \varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i,$$

$$\lambda = m^2 + \hat{\lambda}$$



EFFECTIVE ACTION FOR $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_{\theta}[\hat{\lambda}] = \sum_{i=1}^N \left[v\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \text{Tr} \left[\log \left(-D_{\mu}^i D_{\mu}^i + m^2 + \hat{\lambda} \right) \right] \right].$$

Non-local action for $\hat{\lambda}$.

To be expanded order-by-order in $1/N$.

We can identify the functional determinant with the grand-canonical (fixed chemical potential) free energy:

$$F_{\text{gc}}(i\theta) = \sum_{i=1}^N \left[v \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \frac{1}{\beta} \text{Tr} \left[\log \left(-D_{\mu}^i D_{\mu}^i + m^2 \right) \right] \right].$$



ZETA FUNCTIONS

In the limit $\beta \rightarrow \infty$ (zero temperature), we regularize with a zeta function

$$\zeta(s|\Sigma, m) = \sum_p (E(p)^2 + m^2)^{-s}:$$

The gap equations are (set $A_1 = v$, $A_{>1} = 0$):

$$\frac{\delta}{\delta m} : v v^2 + \frac{N-1}{2} \zeta(1/2|\Sigma, m) = 0,$$

$$\frac{\delta}{\delta \theta} : -iQ + \frac{2V}{\beta} \theta v^2 = 0,$$

$$\frac{\delta}{\delta v} : 2V\beta \left(m^2 + \frac{\theta^2}{\beta^2} \right) v = 0,$$

For finite Q we need necessarily $v \neq 0$ and then $\theta = i m \beta$. So we get

$$m \zeta(1/2|\Sigma, m) = -\frac{Q}{N-1}$$



ORDER N

At leading order in N, the free energy is

$$F(Q) = -\frac{1}{\beta} \left(i\theta Q + N \frac{\partial}{\partial s} \frac{\Gamma(s - 1/2)}{2\sqrt{\pi}\Gamma(s)} \beta \zeta(s - 1/2 | \Sigma, m) \right) \Big|_{s=0}$$

Using the gap equations

$$F(Q) = mQ + N\zeta(-1/2 | \Sigma, m)$$

For $\Sigma = S^2$ at large Q/N :

$$F(Q) = \frac{N\sqrt{2}}{3} \left(\frac{Q}{N} \right)^{3/2} + \frac{N}{3\sqrt{2}} \left(\frac{Q}{N} \right)^{1/2} - \frac{7N}{180\sqrt{2}} \left(\frac{Q}{N} \right)^{-1/2} + \dots$$



SMALL Q/N

The zeta function can be expanded in perturbatively in small Q/N.

Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2} \frac{Q}{N} + \frac{16(\pi^2 - 12)}{3\pi^4 N^2} Q^2 + \dots$$

- Expansion of a closed expression
- It's a convergent series with finite radius of curvature (marks the threshold of the EFT)
- Start with the engineering dimension 1/2
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation



ORDER N

$$F_{S^2}(Q) = \frac{4N}{3} \left(\frac{Q}{2N} \right)^{3/2} + \frac{N}{3} \left(\frac{Q}{2N} \right)^{1/2} \\ - \frac{7N}{360} \left(\frac{Q}{2N} \right)^{-1/2} - \frac{71N}{90720} \left(\frac{Q}{2N} \right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$



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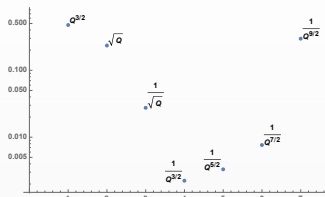


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FINAL RESULT

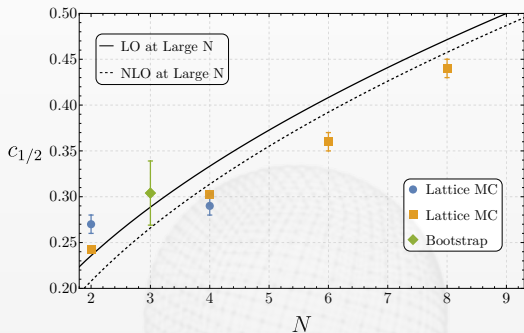
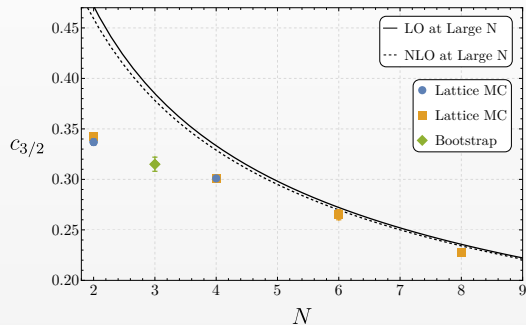
$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(N^0)\right) \left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(N^0)\right) \left(\frac{Q}{2N}\right)^{1/2} + \dots$$

- 0.0937...



FINAL RESULT

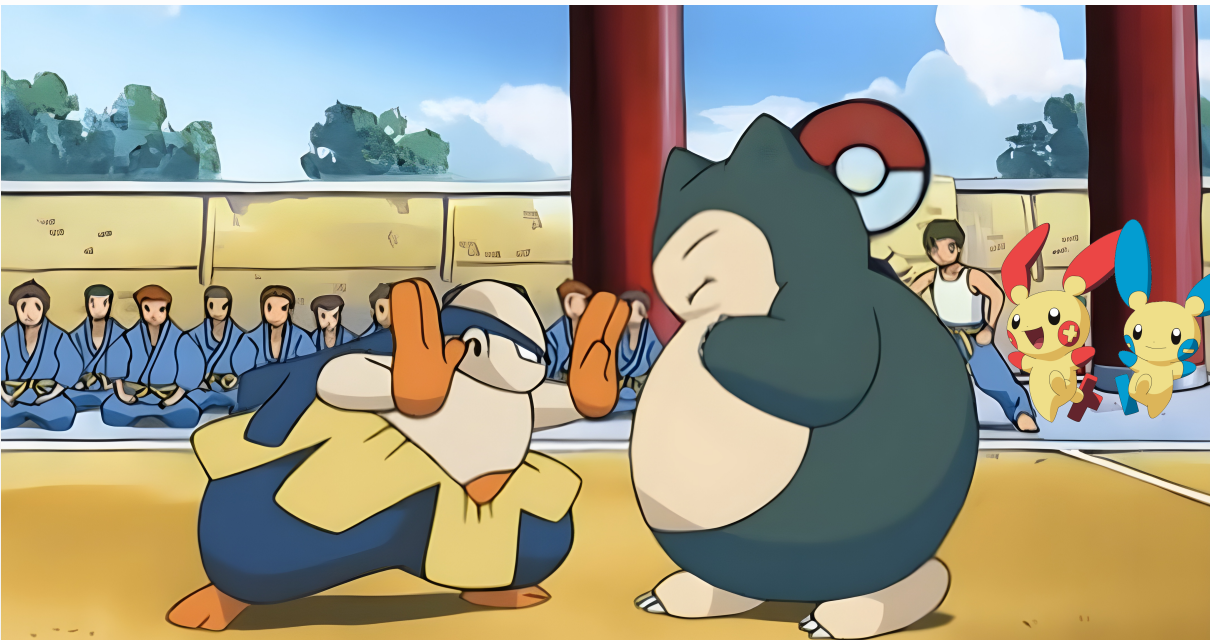
$$\Delta(\mathcal{Q}) = \left(\frac{4N}{3} + \mathcal{O}(N^0) \right) \left(\frac{\mathcal{Q}}{2N} \right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(N^0) \right) \left(\frac{\mathcal{Q}}{2N} \right)^{1/2} + \dots - 0.0937 \dots$$



[Dondi & Sberveglieri, arXiv:2409.06781]



FERMIONS AT LARGE N



GROSS—NEVEU AND NAMBU—JONA—LASINIO

Simplest four-Fermi models

$$S_{\text{GN}}[\psi] = \int dt d\Sigma \sum_{i=1}^N \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{2N} (\bar{\psi} \psi)^2$$

$$S_{\text{NJL}}[\psi, \varphi] = \int dt d\Sigma \sum_{i=1}^N \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{N} \left[(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2 \right]$$

Respectively $O(4N)$ and $U(N) \times U(1)$ symmetry.

Non-trivial CFT in the ultraviolet (UV).

Convenient to introduce collective fields

$$\sigma = \frac{g}{N} \bar{\psi} \psi$$

$$\varphi = \frac{g}{N} (\bar{\psi} \psi + \bar{\psi} \gamma_5 \psi)$$



GNY AND NJL MODELS AT LARGE N

$$S_{\text{GNY}}[\psi, \sigma] = \int dt d\Sigma \sum_{i=1}^N \bar{\psi}_i \left(\gamma^\mu \partial_\mu + \sigma \right) \psi_i + \frac{1}{2g_Y} \partial_\mu \sigma \partial_\mu \sigma.$$

$$S_{\text{NJL}}[\psi, \varphi] = \int dt d\Sigma \sum_{i=1}^N \bar{\psi}_i \left[\gamma^\mu \partial_\mu + \varphi \left(\frac{1+\gamma_5}{2} \right) + \varphi^* \left(\frac{1-\gamma_5}{2} \right) \right] \psi_i + \frac{1}{g_Y} \partial_\mu \varphi^* \partial_\mu \varphi$$

They flow in the IR to the same CFTs.

We want to fix U(1) charges.

$$\psi_i \rightarrow e^{i\alpha} \psi_i,$$

$$\sigma \rightarrow \sigma$$

$$\psi \rightarrow e^{i\alpha\gamma_5} \psi,$$

$$\varphi \rightarrow e^{-2i\alpha} \varphi,$$

and compute the partition function at fixed charge

$$Z(Q) = \text{Tr} \left[e^{-\beta H} \delta(\hat{Q} - Q) \right].$$



FIX THE CHARGE

$$Z_{\Sigma}(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta Q} \text{Tr} \left[e^{-\beta H} e^{-i\theta \hat{Q}} \right].$$

At large N the integral over θ becomes a Legendre transform

$$\Delta(Q) = -\frac{1}{\beta} \log(Z_{S^2}(Q)) = \sup_{i\theta} (i\theta Q - S_{\text{eff}}(\theta))$$

The trace is written as a path integral in a flat connection

$$Z_{\Sigma}(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\substack{\psi(2\pi\beta) = -\psi(0) \\ \varphi(2\pi\beta) = \varphi(0)}} \mathcal{D}\varphi_i e^{-S^{\theta}[\varphi]}$$



EFFECTIVE ACTION: COVARIANT DERIVATIVE

The actions are quadratic in the fermions: Gaussian integral.

For the bosonic fields, we expand as VEV plus fluctuations

$$\sigma = \sigma_0 + \frac{1}{\sqrt{N}} \hat{\sigma}$$
$$\varphi = \Phi_0 + \frac{1}{\sqrt{N}} \hat{\varphi}$$

The leading contribution comes from the VEV:

$$\Omega = S_{\text{eff}} = -N \text{Tr} \log \left(\gamma^\mu \partial_\mu - i \frac{\theta}{\beta} \gamma_3 + \sigma_0 \right)$$
$$\Omega = S_{\text{eff}} = -N \text{Tr} \log \left(\gamma^\mu \partial_\mu - i \frac{\theta}{\beta} \gamma_3 \gamma_5 + \Phi_0 \left(\frac{1+\gamma_5}{2} \right) + \Phi_0^* \left(\frac{1-\gamma_5}{2} \right) \right)$$

Gap equation: minimized wrt σ_0 or Φ_0 (gap equation).

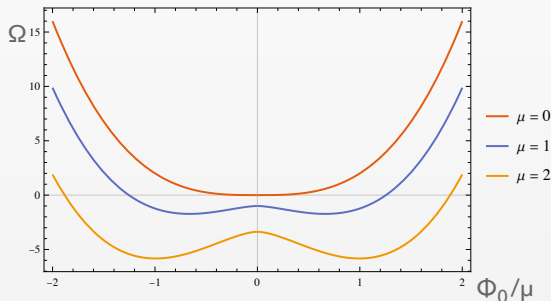
As before, $i\theta/\beta = \mu$ is the chemical potential.



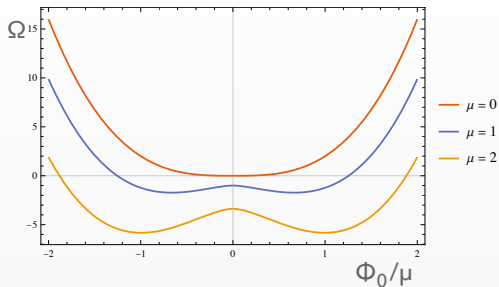
THE NAMBU—JONA—LASINIO MODEL

In the $\beta \rightarrow \infty$ limit, on the torus $\Sigma = \mathbb{T}^2$, the grand potential is

$$\begin{aligned}\frac{\Omega}{N} &= - \int \frac{d^2 p}{(2\pi)^2} \left[\sqrt{(|p| + \mu)^2 + |\Phi_0|^2} + \sqrt{(|p| - \mu)^2 + |\Phi_0|^2} \right] \\ &= - \frac{1}{6\pi} \left[3|\Phi_0|^2 \mu \operatorname{arctanh} \frac{\mu}{\sqrt{|\Phi_0|^2 + \mu^2}} + (\mu^2 - 2|\Phi_0|^2) \sqrt{|\Phi_0|^2 + \mu^2} \right]\end{aligned}$$



THE NAMBU–JONA–LASINIO MODEL



There is a minimum (gap equation) for every value of μ

$$\Phi_0 = \mu \sqrt{\kappa_0^2 - 1} = 0.6627 \dots \mu$$

Same physics: fixing the charge induces a spontaneous symmetry breaking. The field φ is the order parameter for the **superfluid phase transition**.

There is only **one difference**.

For bosons, zero modes fix the gap as function of the chemical potential $m = \mu$. Fermions don't have zero modes and the gap is fixed by a one-loop term.



COOPER PAIRS

@NJL

The scalar φ is a composite field

$$\varphi = \bar{\psi}\psi + \bar{\psi}\gamma^5\psi$$

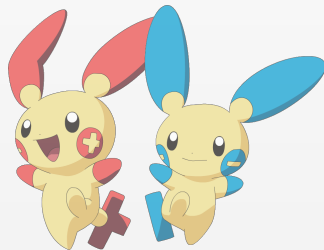
More transparent after a Pauli-Gürsey transformation:

$$\psi \mapsto \frac{1}{2} \left[(1 - \gamma_5)\psi - (1 + \gamma_5)C\bar{\psi}^T \right], \quad \bar{\psi} \mapsto \frac{1}{2} \left[\bar{\psi}(1 + \gamma_5) - \psi^T C(1 - \gamma_5) \right].$$

because then we identify φ with a **Cooper pair**

$$\varphi = \psi^t C \psi$$

In presence of an attractive interactions, fermion form pairs that behave as bosons and undergo a Bose-Einstein transition.



Repeat the computation on S^2 .

Infinite sum to regularize: zeta function for the Dirac operator.

The **action at the saddle** is

$$S(\mu, \Phi_0) = -\beta N \zeta(-1/2 | \partial, \mu, \Phi_0),$$

$$\zeta(s | \partial, \mu, \Phi_0) = 2r^{2s} \sum_{\ell=1}^{\infty} \ell [((\ell + r\mu)^2 + (r\Phi_0)^2)^{-s} + ((\ell - r\mu)^2 + (r\Phi_0)^2)^{-s}].$$

1. Minimize with respect to Φ_0 for the gap equation.
2. Legendre transform from the chemical potential μ to the charge $q = Q/(2N)$.



SMALL CHARGE

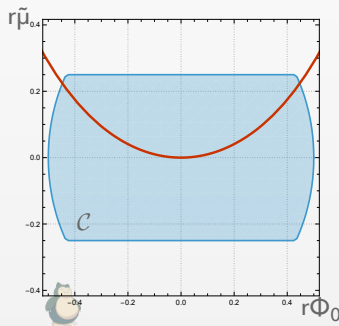
@NJL

$$\frac{S(\tilde{\mu}, \Phi_0)}{\beta N} = -\frac{\pi^2 r^2 \tilde{\mu}}{8} (\Phi_0^2 + \tilde{\mu}^2) - \frac{3\pi^2 r^3 \tilde{\mu}^2}{8} (\Phi_0^2 + \tilde{\mu}^2) + \frac{\pi^2 r^3}{16} (\Phi_0^2 + \tilde{\mu}^2)^2 + \dots$$

And the gap equation is

$$r\tilde{\mu} = (r\Phi_0)^2 + \frac{\pi^2 - 8}{4} (r\Phi_0)^4 + \frac{192 - 8\pi^2 - \pi^4}{24} (r\Phi_0)^6 + \dots,$$

Finite region of convergence



1. invert the gap equation and write the gap Φ_0 as function of the chemical potential μ ,
2. Legendre transform from μ to $q = Q/(2N)$

Conformal dimensions in the small q limit

$$\frac{\Delta}{2N} = \frac{1}{2}q + \frac{2}{n^2}q^2 + \frac{8(n^2 - 12)}{3n^4}q^3 + \frac{2(n^2 - 48)(3n^2 - 32)}{3n^6}q^4 + \dots$$

Consistent with $\Delta(Q) \approx Q/2$.

Once more, it is the resummation of an infinite number of diagrams.

The series can be computed at any given order, and it has a finite radius of convergence $|q| \leq 0.35(3)$.



Use a Mellin representation (heat kernel)

$$\zeta(s|\phi, \mu, \Phi_0) = \frac{2}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\Phi_0^2 t} \sum_{\ell=1}^\infty \ell (e^{-(\ell/r+\mu)^2 t} + e^{-(\ell/r-\mu)^2 t}) .$$

Action at the saddle

$$\begin{aligned} \frac{S(\mu, \Phi_0)}{\beta N} = & \left(\frac{\mu^2 - 2\Phi_0^2}{3} \sqrt{\mu^2 + \Phi_0^2} + \mu\Phi_0^2 \operatorname{arcsinh}\left(\frac{\mu}{\Phi_0}\right) \right) r^2 + \frac{1}{6} \sqrt{\mu^2 + \Phi_0^2} \\ & + \frac{\Phi_0^2}{120 (\mu^2 + \Phi_0^2)^{3/2}} r^2 + \frac{\Phi_0^4 - 4\mu^2 \Phi_0^2}{1008 (\mu^2 + \Phi_0^2)^{7/2}} r^4 + \dots \end{aligned}$$



1. invert the gap equation and write the gap Φ_0 as function of the chemical potential μ ,
2. Legendre transform from μ to Q/N



$$F_{S^2}(Q) = \frac{4N}{3} \left(\frac{Q}{2N\kappa_0} \right)^{3/2} + \frac{N}{3} \left(\frac{Q}{2N\kappa_0} \right)^{1/2} \\ - \frac{11 - 6\kappa_0^2}{360\kappa_0^2} \left(\frac{Q}{2N\kappa_0} \right)^{-1/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$



ORDER N

@NJL



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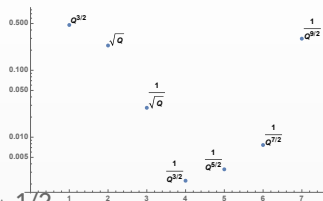
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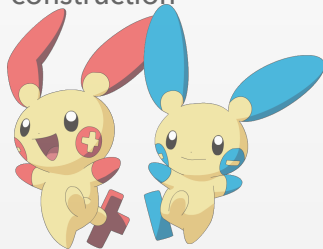
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WHAT HAS HAPPENED?

@NJL

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge results in **spontaneous symmetry breaking**
- The order parameter is a composite field (**Cooper pair**)
- We compute the conformal dimension of the lowest operator of charge Q
- We find a result in **total agreement** with the general EFT construction



THE GROSS-NEVEU-YUKAWA MODEL

Now for the Gross-Neveu-Yukawa model.

Integrating out the Matsubara frequencies we find

$$\frac{\Omega}{N} = -2 \int \frac{d^2 p}{(2\pi)^2} \left\{ \omega_p + \frac{1}{\beta} \log \left(1 + e^{-\beta(\omega_p + \mu)} \right) + (\mu \leftrightarrow -\mu) \right\},$$

with $\omega_p^2 = p^2 + \sigma_0^2$. The gap equation is

$$\sigma_0 - \frac{1}{\beta} \log \left((1 + e^{\beta(\sigma_0 + \mu)}) (1 + e^{\beta(\sigma_0 - \mu)}) \right) = 0$$

and admits only the solution $\sigma_0 = 0$.

In other words, in the large- N limit the symmetry is never broken.

This is **different** from the superfluid EFT behavior.



THE GROSS–NEVEU–YUKAWA MODEL

We can repeat the computation on the sphere. The grand potential is

$$\frac{\Omega}{N} = -\frac{1}{2\pi r_0^2} \left[\sum_{\omega_j > \mu} (2j+1)\omega_j + \mu \sum_{\omega_j < \mu} (2j+1) \right]$$

the solution to the gap equation and the Legendre transform are

$$\frac{Q}{N} = \frac{1}{2\pi r_0^2} \lfloor \mu r_0 \rfloor (\lfloor \mu r_0 \rfloor + 1), \quad \frac{E}{N} = \frac{1}{6\pi r_0^3} \lfloor \mu r_0 \rfloor (\lfloor \mu r_0 \rfloor + 1) (2\lfloor \mu r_0 \rfloor + 1).$$

This is the physics of a Fermi sphere.

However, the behavior of the conformal dimension is still the same

$$\Delta = \frac{2}{3} \left(\frac{Q}{2N} \right)^{3/2} + \frac{1}{12} \left(\frac{Q}{2N} \right)^{1/2} - \frac{1}{192} \left(\frac{Q}{2N} \right)^{-1/2} + \dots$$



WHAT HAS HAPPENED?

@GNY

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge **does not result** in spontaneous symmetry breaking
- The physics is the one of a **Fermi sphere**
- The dimension of the lowest operator still obeys the same law
- The Fermi sphere is unstable if there is an attractive channel: there must be a finite-N transition and an **exponentially-suppressed gap**



RESURGENCE AND THE LARGE CHARGE



RESULTS FROM LARGE N

$O(2N)$ at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \rightarrow \infty$, $Q \rightarrow \infty$ with $\hat{q} = Q/(2N)$ fixed.

$$\begin{cases} F_{\Sigma}^{\text{web}}(Q) = \mu Q + N\zeta(-\frac{1}{2}|\Sigma, \mu), \\ \mu\zeta(\frac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}. \end{cases}$$



RESULTS FROM LARGE N

$O(2N)$ at criticality in $1 + 2$ dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \rightarrow \infty$, $Q \rightarrow \infty$ with $\hat{q} = Q/(2N)$ fixed.

The free energy per DOF $f(\hat{q}) = F/(2N)$ is

$$f(\hat{q}) = \sup_{\mu} (\mu \hat{q} - \omega(\mu)),$$

$$\omega(\mu) = -\frac{1}{2} \zeta(-\frac{1}{2} | \Sigma, \mu),$$



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$\zeta(s | \Sigma, \mu)$ is the zeta function for the operator $-\Delta + \mu^2$. In Mellin representation

$$\zeta(s | \Sigma, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t} \text{Tr}(e^{\Delta t}).$$



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Large $\hat{q}$ is large μ and is small t . The classical Seeley-de Witt problem:

$$\text{Tr}(e^{\Delta t}) \sim \frac{V}{4\pi t} \left(1 + \frac{R}{12} t + \dots \right).$$



THE TORUS

As a warm-up: $\Sigma = \mathbb{T}^2$.

$$\text{spec}(\Delta) = \left\{ -\frac{4\pi^2}{L^2} (k_1^2 + k_2^2) \mid k_1, k_2 \in \mathbb{Z} \right\}.$$

It follows that the heat kernel trace is the square of a theta function:

$$\text{Tr}(e^{\Delta t}) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2} (k_1^2 + k_2^2)t} = \left[\theta_3(0, e^{-\frac{4\pi^2 t}{L^2}}) \right]^2.$$

We are interested in the small- t limit: we Poisson-resum the series:

$$\text{Tr}(e^{\Delta t}) = \left[\frac{L}{\sqrt{4\pi t}} \left(1 + \sum_{k \in \mathbb{Z}} e^{-\frac{k^2 L^2}{4t}} \right) \right]^2 = \frac{L^2}{4\pi t} \left(1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{\|k\|^2 L^2}{4t}} \right)$$



THE TORUS

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\frac{1}{2}|T^2, \mu) = \frac{L^2\mu^3}{12\pi} \left(1 + \sum_k \frac{e^{-\|k\|\mu L}}{\|k\|^2\mu^2L^2} \left(1 + \frac{1}{\|k\|\mu L} \right) \right).$$

Free energy

$$f(\hat{q}) = \sup_{\mu} (\mu\hat{q} - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L} \hat{q}^{3/2} \left(1 - \sum_k \frac{e^{-\|k\|\sqrt{4\pi\hat{q}}}}{8\|k\|^2\pi\hat{q}} + \dots \right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/\hat{q}$ and $e^{-\sqrt{4\pi\hat{q}}}$.
- non-perturbative effects more important than the “usual” instantons $\mathcal{O}(e^{-\hat{q}})$



THE SPHERE

On the two sphere $\text{spec}(\Delta) = \{-\ell(\ell + 1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell + 1$.

Again, we use Poisson resummation

$$\text{Tr}(e^{\Delta t})e^{-t/4} = \sum_{\ell \geq 0} (2\ell + 1)e^{-(\ell+1/2)^2 t} \sim \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (1 - 2^{1-2n})}{n!} B_{2n} t^n$$

The series is asymptotic: the Seeley-de Witt coefficients diverge like $n!$:

$$a_n = \frac{(-1)^{n+1} (1 - 2^{1-2n})}{n!} B_{2n} \sim \frac{2n^{1/2}}{n^{5/2+2n}} n!.$$

this divergence is reflected in the existence of non-perturbative corrections.



BOREL RESUMMATION



gettyimages
ulstein bild Dtl.

BOREL TRANSFORM

We need to make sense of the divergent series and the imaginary terms.


$$H(t) = \sum_{n \geq 0} a_n t^n \xrightarrow{\quad} \hat{H}(\tau) = \sum_{n \geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

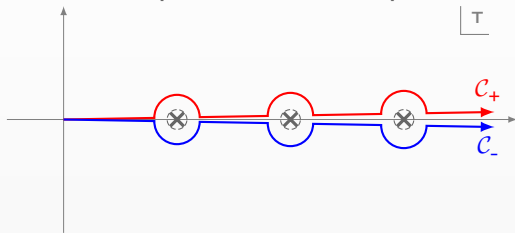
$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(tw^\beta) \frac{dw}{w}$$

The diagram illustrates the Borel transform process. A solid arrow points from the divergent series $H(t) = \sum_{n \geq 0} a_n t^n$ to the Borel transform $\hat{H}(\tau) = \sum_{n \geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$. A dashed arrow points from $\hat{H}(\tau)$ to the Borel sum $s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(tw^\beta) \frac{dw}{w}$. A solid arrow also points from $\hat{H}(\tau)$ back to $H(t)$, indicating the inverse transform.



LATERAL TRANSFORM

If there are poles on the real positive axis there is an ambiguity



$$s_{\pm}(H)(t) = s(H)(t) = \int_{C_{\pm}} w^b e^{-w} \hat{H}(tw^{\beta}) \frac{dw}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.



MORE INGREDIENTS



WORLDLINE INTERPRETATION

We need a **non-perturbative interpretation** of these exponential terms.

We read the heat kernel as the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \Delta$, i.e. a **free quantum particle moving on $\mathbb{R} \times \Sigma$** .

We can write the partition function as a **path integral**

$$\mathrm{Tr}\left(e^{(\partial_0^2 + \Delta)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t} \int_0^1 d\tau g_{\mu\nu} \dot{X}^\mu(\tau) \dot{X}^\nu(\tau)$$



A TRANSERIES FROM GEODESICS

In the limit $t \rightarrow 0$ the path integral localizes on a sum over all the closed geodesics γ .

For each geodesic a perturbative series in t , weighted by $e^{-\ell(\gamma)^2/(4t)}$

$$\begin{aligned}\mathrm{Tr}\left(e^{(\partial_0^2 + \Delta)t}\right) &= \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X e^{-S[X]} \\ &= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma \in \text{closed geodesics}} e^{-\frac{\ell(\gamma)^2}{4t}} t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,\end{aligned}$$

the b_γ depend on the geometry.

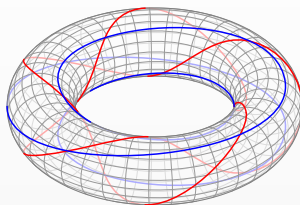
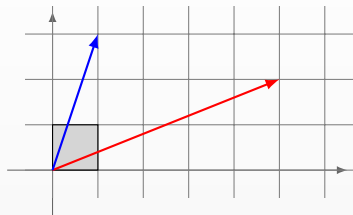
This is precisely the same structure predicted by resurgence.

Now we have a geometric interpretation.



THE TORUS

In the case of the torus, closed geodesics are labelled by two integers (k_1, k_2)



The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h e^{-\frac{1}{4t} \int_0^1 d\tau (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t} \partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$



THE TORUS

Now we can write the result of the path integral

$$\begin{aligned}\mathrm{Tr}(e^{\Delta t}) &= \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{\mathrm{cl}} h(1)=h(0)=0} \int e^{-S[X_{\mathrm{cl}}]-S[h]} \\ &= \mathcal{N} L^2 \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2(k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h e^{-S[h]}, \\ &= \frac{L^2}{4\pi t} \left[1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 \|k\|^2}{4t}} \right]\end{aligned}$$

This is exactly what we had found before just by looking at the spectrum. Now we can understand the non-perturbative effects in terms of closed geodesics.



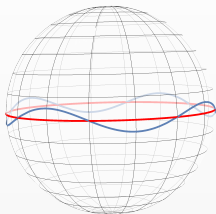
THE SPHERE

Closed geodesics on the sphere go around the equator k times



THE SPHERE

Closed geodesics on the sphere go around the equator k times

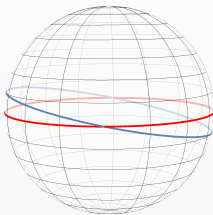
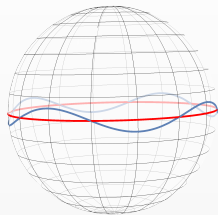


We need to sum over the fluctuations h_φ and h_θ



THE SPHERE

Closed geodesics on the sphere go around the equator k times



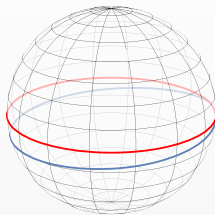
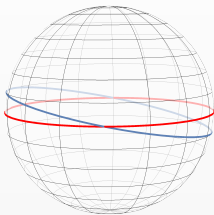
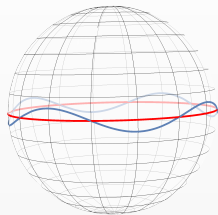
We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator



THE SPHERE

Closed geodesics on the sphere go around the equator k times



We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator

And an instability because we can slide off



BACK TO RESURGENCE

Putting it **all together**, the non-trivial geodesics give

$$\pm 2i \left(\frac{n}{t} \right)^{3/2} \sum_{k \in \mathbb{Z}} |k| e^{-\frac{k^2 n^2}{t}}$$

The one-loop result **perfectly cancels** the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr} \left(e^{(\Delta - \frac{1}{4})t} \right) = s_{\pm}(H)(t) \mp 2i \left(\frac{n}{t} \right)^{3/2} \sum_{k \geq 1} (-1)^k k e^{-\frac{k^2 n^2}{t}} = \mathrm{Re}[s_{\pm}(H)(t)]$$



BACK TO RESURGENCE

We can write the **exact expression** for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \text{Re} \left[\frac{2rm^2}{\pi} \int_0^\infty dy \frac{K_2(2mry)}{y \sin(y)} \right] = \frac{r^2}{3} m^3 - \frac{m}{24} + \dots - \frac{2ir^{1/2} m^{3/2}}{(4\pi)^{3/2}} e^{-2\pi rm} + \dots$$



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As a numerical test, we can compare with the convergent small-charge expansion ($\hat{q} \approx 0.6$)

$$r\omega(mr = 0.4) \Big|_{\text{small charge}} = 0.012\,777\,296\,63\dots$$

$$r\omega(mr = 0.4) \Big|_{\text{resurgence}} = 0.012\,777\,297\,69\dots$$



OPTIMAL TRUNCATION



LESSONS FROM LARGE N

Let's go back to the EFT.

The effective action is identified with the asymptotic expansion: the **grand potential** is the value of the **action at the minimum** $\chi = \mu t$:

$$\omega(\mu) = L_{\text{EFT}} \Big|_{\chi=\mu t}$$

where

$$L_{\text{EFT}} = \omega_0 (\partial_\mu \chi \partial^\mu \chi)^{3/2} + \omega_1 (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots,$$

In general the **coefficients are unknown**

BUT

Now we have a **geometric understanding** of the non-perturbative effects



LESSONS FROM LARGE N

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is
a particle of mass μ going around the equator.

A CFT has no intrinsic scales.

The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n \geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi k f_0^{(0)} \sqrt{Q}} \sum_{n \geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \dots$$

(we used $\mu = 3f_0^{(0)} \sqrt{Q}/2 + \dots$)



LESSONS FROM LARGE N

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LESSONS FROM LARGE N

- The **controlling parameter** for the non-perturbative effects $e^{-3\pi k f_0^{(0)} \sqrt{Q}}$ is fixed by the **leading term** in the $1/Q$ expansion.
- The non-perturbative coefficient $e^{-3\pi k f_0^{(0)} \sqrt{Q}}$ fixes the **large-n behavior** of the perturbative series $f_n^{(0)}$.

$$f_n^{(0)} \sim (2n)!(3\pi k f_0^{(0)})^{-n}$$

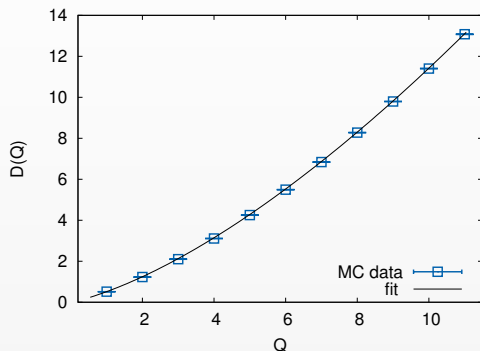
We don't know enough for a Borel resummation, but we can estimate an optimal truncation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi k f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$



CAN WE UNDERSTAND THE LATTICE RESULTS NOW?



In $O(2)$, $f_0^{(0)} \approx 0.301(3)$

so $N^* = \mathcal{O}(\sqrt{Q})$ and $\varepsilon(Q) = \mathcal{O}(e^{-n\sqrt{Q}})$.

Lattice:

Best fit with $N = 3$ terms.

At $Q = 1$ the error is $\approx 6 \times 10^{-2}$; at $Q = 11$ the error is $\approx 5 \times 10^{-5}$.

Resurgence:

$\sqrt{10} \approx 3.16$

$e^{-n} \approx 4 \times 10^{-2}$ and $e^{-n\sqrt{11}} = 3 \times 10^{-5}$.



WHAT HAS HAPPENED?

- The large-charge expansion of the Wilson-Fisher point is **asymptotic**
- In the **double-scaling** limit $Q \rightarrow \infty, N \rightarrow \infty$ we control the perturbative expansion
- We can **Borel**-resum the expansion
- We have a **geometric interpretation for the non-perturbative effects**
- We can use this geometric interpretation also in the **finite-N** case
- We obtain an **optimal truncation** and estimate of the error
- The results are **consistent with lattice simulations**



CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Precise and **testable predictions**.
- Qual(nt)itative control of the **non-perturbative** effects.
- **CFT constraints**: perturbative/non-perturbative **interplay**.
- Remarkable agreement with **lattice**.



AN EFT FOR A CFT

USE THE SYMMETRY



YOU MUST

THE O(2) MODEL

The simplest example is the WF point of the O(2) model in three dimensions.

- Non-trivial fixed point of the φ^4 action

$$L_{UV} = \partial_\mu \varphi^* \partial_\mu \varphi - u(\varphi^* \varphi)^2$$

- Strongly coupled
- In nature: ${}^4\text{He}$.
- Simplest example of spontaneous symmetry breaking.
- **Not accessible** in perturbation theory. **Not accessible** in $4 - \varepsilon$. **Not accessible** in large N.
- Lattice. Bootstrap.

CHARGE FIXING

We consider a **subsector of fixed charge Q** .

Generically, the classical solution at fixed charge **breaks spontaneously**
 $U(1) \rightarrow \emptyset$.

We have one **Goldstone boson χ** .

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_n}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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$$L[\chi] = \frac{f_n}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

We want to describe a CFT: we can **dress with a dilaton**

$$L[\sigma, \chi] = \frac{f_n e^{-2f\sigma}}{2} \partial_\mu \chi \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken $U(1)$, the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

LINEAR SIGMA MODEL

We can put together the two fields as

$$\Sigma = \sigma + if_n \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_n$ and $u = 3(Cf^2)^3$.

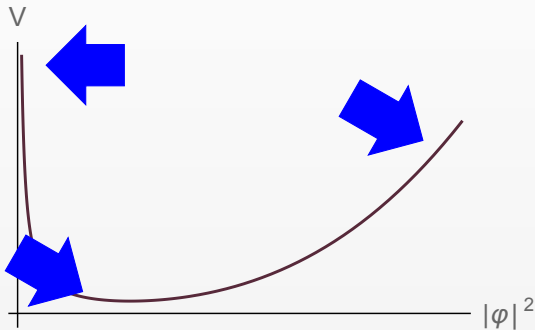
Scale invariance is manifest.

The field φ is some complicated function of the original φ .

CENTRIFUGAL BARRIER

The $O(2)$ symmetry acts as a shift on χ .

Fixing the charge is the same as adding a **centrifugal term** $\propto \frac{1}{|\varphi|^2}$.



GROUND STATE

We can find a fixed-charge solution of the type

$$\chi(t, x) = \mu t \qquad \sigma(t, x) = \frac{1}{f} \log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \dots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2} / \sqrt{V} Q^{3/2} + c_{1/2} R \sqrt{V} Q^{1/2} + \mathcal{O}(Q^{-1/2})$$

FLUCTUATIONS

The fluctuations over this ground state are described by two modes.

- A universal “**conformal Goldstone**”. It comes from the breaking of the $U(1)$.

$$\omega = \frac{1}{\sqrt{2}}p$$

- The **massive dilaton**. It controls the magnitude of the quantum fluctuations.
All quantum effects are controlled by $1/Q$.

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

NON-LINEAR SIGMA MODEL

Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2} (\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.

All other terms are suppressed by powers of $1/Q$.

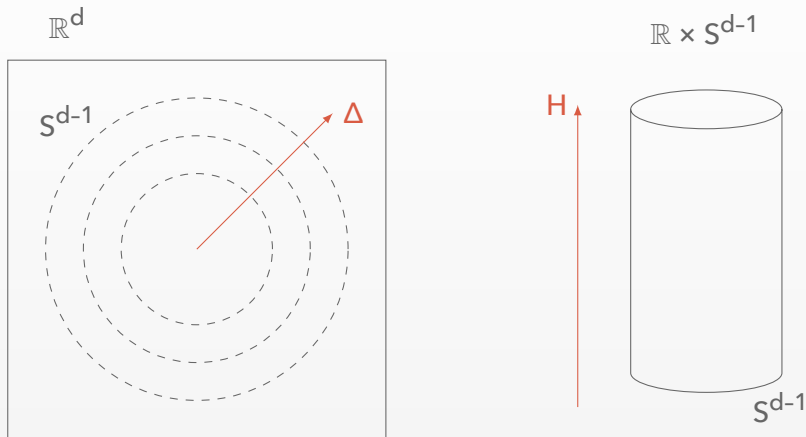
In 3 + 1 NRCFT the analogous story in a background potential A_0 leads to

$$L[\chi] = c_0 U^{5/2} + c_1 U^{-1/2} \partial_i U \partial_i U + c_2 U^{1/2} ((\partial_i \partial_i \chi)^2 - 9 \partial_i \partial_i A_0) + \dots \quad (1)$$

where $U = \partial_t \chi - A_0 \chi - \partial_i \chi \partial_i \chi / 2$.

STATE-OPERATOR CORRESPONDENCE

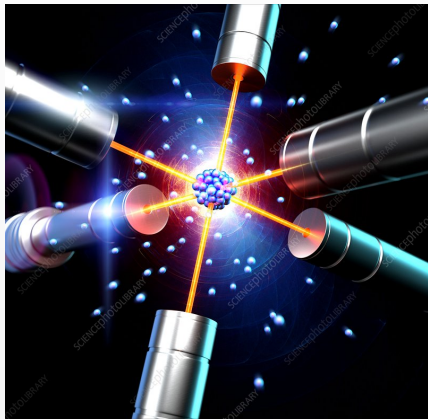
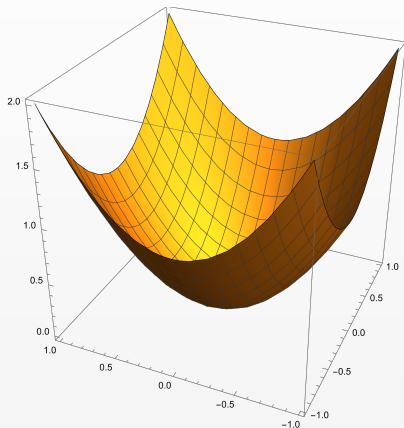
The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

NRCFT STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in a harmonic trap.



Protected by conformal invariance: a well-defined quantity.

CONFORMAL DIMENSIONS

We know the energy of the ground state.

The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}} \zeta(-\frac{1}{2}|S^2) = -0.0937...$$

This is the unique contribution of order Q^0 .

Final result: the **conformal dimension of the lowest operator of charge Q** in the $O(2)$ model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094... + \mathcal{O}(Q^{-1/2})$$

In $3+1$ NRCFT we find

$$\Delta_Q = c_{4/3} Q^{4/3} + c_{2/3} Q^{2/3} + b_{5/9} Q^{5/9} + b_{1/3} Q^{1/3} + b_{1/9} Q^{1/9} - \frac{1}{3\sqrt{3}} \log(Q) + c_0$$

WHAT HAPPENED?

We started from a CFT.

There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple EFT**.

We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

ORDER N^0

The order N^0 terms are

$$S^0[\hat{\sigma}, \hat{\lambda}] = \int dt d\Sigma \left((D_\mu \hat{\sigma})^* (D^\mu \hat{\sigma}) + (\mu^2 + \hat{\lambda}) \hat{\sigma}^* \hat{\sigma} + \frac{\hat{\lambda}_V (\hat{\sigma} + \hat{\sigma}^*)}{(N-1)^{1/2}} \right) \\ + \frac{1}{2} \int dx_1 dx_2 \hat{\lambda}(x_1) \hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.

At low energies we can approximate the non-local term as

$$\int dt d\Sigma \hat{\lambda}(x)^2 \zeta(2|\theta, \Sigma, \mu) \approx \frac{V}{2\mu} \int dt d\Sigma \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

ORDER N°

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4\mu^2) & \mu\omega \\ -\mu\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \dots = 0$$

$$\omega^2 + 8\mu^2 + \frac{3}{2}p^2 + \dots = 0$$

This is the conformal Goldstone that we have seen in the EFT.

Its contribution to the partition function is

$$E_G = \frac{1}{2} \frac{1}{\sqrt{2}} \zeta(1/2|S^2) = -0.0937\dots$$

This is **universal**. Does not depend on N or Q.

HIGHER ORDERS

There are infinite non-local terms

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx_1 \dots dx_n \hat{\lambda}(x_1) \dots \hat{\lambda}(x_n) P(x_1, \dots, x_n)$$

At low energy they are approximated by

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx \hat{\lambda}(x)^n C_n$$

HIGHER ORDERS

There is only one scale, the charge density $\rho = Q/V$. We must have

$$C_n = \rho^{3/2-n} C_n$$

So

$$S_{nl} = Q^{3/2} \sum_{n=3}^{\infty} \frac{C_n}{n(N-1)^{n/2-1}} \int dx \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by $1/N$.