The Resurgence of the Large Charge expansion

Domenico Orlando
INFN | Torino

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WHAT HAPPENED?

We started from a conformal field theory (CFT).
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.
In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

would you like to know more?

TODAY'S TALK

Justify and prove all these claims from first principles

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Justify and prove all these claims from first principles

Use resurgence for the large-charge EFT

TODAY'S TALK

Justify and prove all these claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

Use resurgence for the large-charge EFT

TODAY'S TALK

Justify and prove all these claims from first principles

Use resurgence for the large-charge EFT

- Borel resum the double-scaling $Q \to \infty$, $N \to \infty$ limit
- geometric interpretation of non-perturbative effects
- general structure of the corrections in the EFT

LARGE N VS. LARGE CHARGE

THE MODEL

φ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int dt\, d\Sigma \left[g^{\mu\nu} \left(\partial_\mu \varphi_i\right)^* \left(\partial_\nu \varphi_i\right) + r \varphi_i^* \varphi_i + \frac{u}{2} \left(\varphi_i^* \varphi_i\right)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.

We compute the partition function at fixed charge

$$Z(Q_1, \ldots, Q_N) = \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int d\Sigma\, j_i^0 = i \int d\Sigma \left[\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i \right].$$

FIX THE CHARGE

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i}\, \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$Z_{\Sigma}(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i\, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i\, e^{-S^{\theta}[\varphi]}$$

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EFFECTIVE ACTION: COVARIANT DERIVATIVE

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int dt\, d\Sigma \left((D_{\mu}\varphi_i)^{*} (D^{\mu}\varphi_i) + \frac{R}{8}\varphi_i^{*}\varphi_i + 2u(\varphi_i^{*}\varphi_i)^2 \right)$$

$$\begin{cases} D_0\varphi = \partial_0\varphi + i\dfrac{\theta}{\beta}\varphi \\ D_i\varphi = \partial_i\varphi \end{cases}$$

Stratonovich transformation

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int dt\, d\Sigma \left[\left(D_{\mu}^{i}\varphi_i\right)^{*} \left(D_{\mu}^{i}\varphi_i\right) + (r+\lambda)\varphi_i^{*}\varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}}A_i + u_i, \qquad \lambda = m^2 + \hat{\lambda}$$

EFFECTIVE ACTION FOR $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_\theta[\hat{\lambda}] = \sum_{i=1}^{N} \left[V\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 + \hat{\lambda} \right) \right] \right].$$

Non-local action for $\hat{\lambda}$.

To be expanded order-by-order in 1/N.

We can identify the functional determinant with the grand-canonical (fixed chemical potential) free energy:

$$F_{gc}(i\theta) = \sum_{i=1}^{N} \left[V \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \frac{1}{\beta} \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 \right) \right] \right].$$

ZETA FUNCTIONS

In the limit $\beta \to \infty$ (zero temperature), we regularize with a zeta function $\zeta(s|\Sigma, m) = \sum_p (E(p)^2 + m^2)^{-s}$:

The gap equations are (set $A_1 = v$, $A_{>1} = 0$):

$$\frac{\delta}{\delta m} : Vv^2 + \frac{N-1}{2}\zeta(1/2|\Sigma, m) = 0,$$

$$\frac{\delta}{\delta \theta} : -iQ + \frac{2V}{\beta}\theta v^2 = 0,$$

$$\frac{\delta}{\delta v} : 2V\beta\left(m^2 + \frac{\theta^2}{\beta^2}\right)v = 0,$$

For finite Q we need necessarily $v \neq 0$ and then $\theta = im\beta$. So we get

$$m\zeta(1/2|\Sigma, m) = -\frac{Q}{N-1}$$

ORDER N

At leading order in N, the free energy is

$$F(Q) = -\frac{1}{\beta}\left(i\theta Q + N \frac{\partial}{\partial s} \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \beta\zeta(s-1/2|\Sigma, m)\Big|_{s=0} \right)$$

Using the gap equations

$$F(Q) = mQ + N\zeta(-1/2|\Sigma, m)$$

For $\Sigma = S^2$ at large Q/N:

$$F(Q) = \frac{N\sqrt{2}}{3}\left(\frac{Q}{N}\right)^{3/2} + \frac{N}{3\sqrt{2}}\left(\frac{Q}{N}\right)^{1/2} - \frac{7N}{180\sqrt{2}}\left(\frac{Q}{N}\right)^{-1/2} + \dots$$

SMALL Q/N

The zeta function can be expanded in perturbatively in small Q/N.
Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2}\frac{Q}{N} + \frac{16\left(\pi^2 - 12\right)Q^2}{3\pi^4 N^2} + \dots$$

- Expansion of a closed expression
- Start with the engineering dimension 1/2
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation

ORDER N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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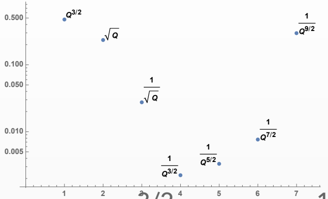
ORDER N



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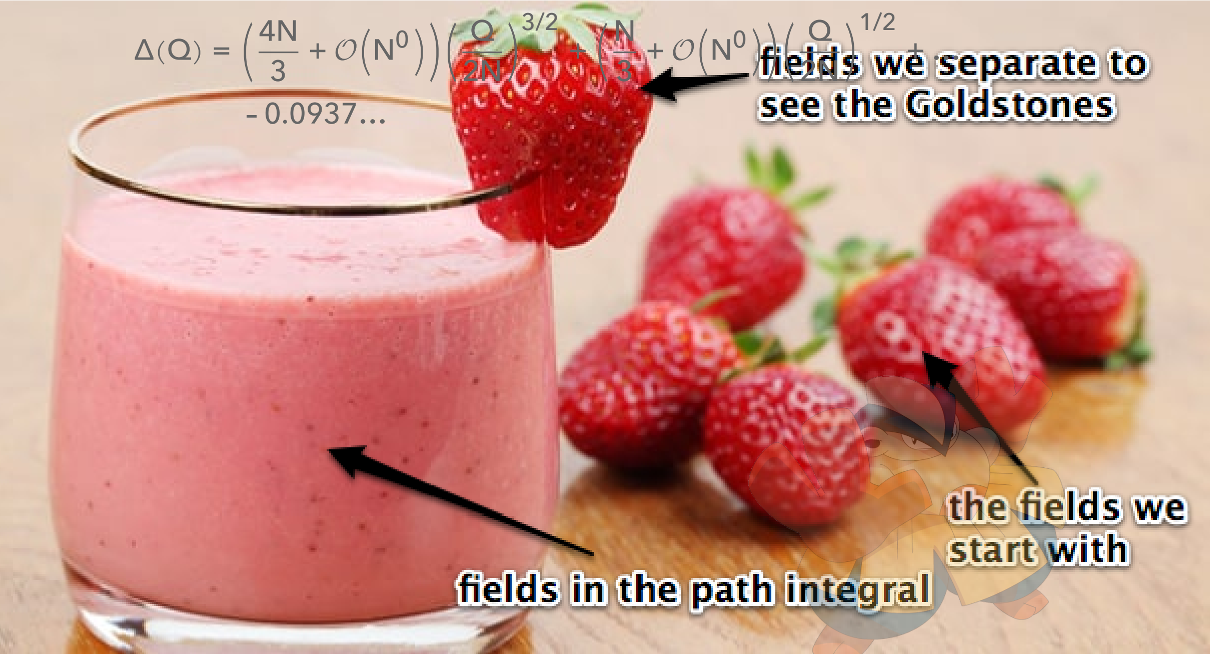
FINAL RESULT

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(N^0)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(N^0)\right)\left(\frac{Q}{2N}\right)^{1/2} + \dots$$

$$- 0.0937\dots$$



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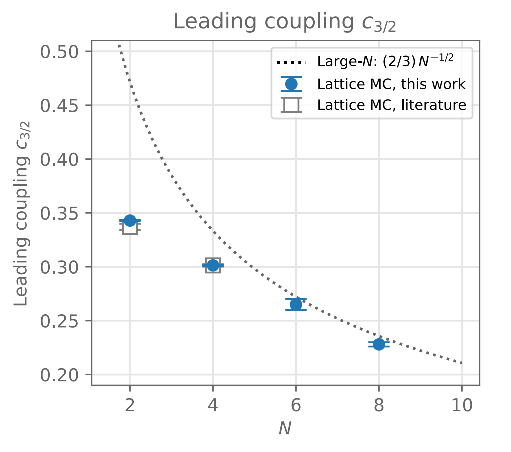
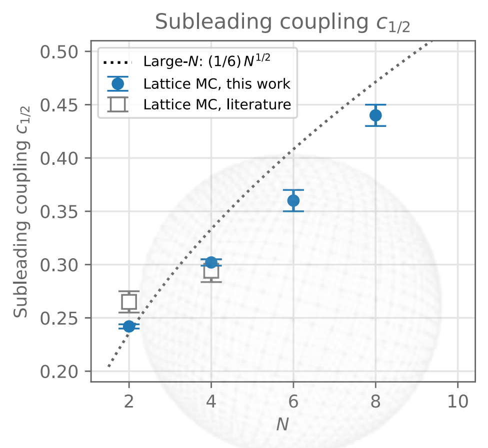
$$- 0.0937\dots$$

fields we separate to see the Goldstones





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FERMIONS AT LARGE N

GROSS—NEVEU AND NAMBU—JONA—LASINIO

Simplest four-Fermi models

$$S_{GN}[\psi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{2N}(\bar{\psi}\psi)^2$$

$$S_{NJL}[\psi, \varphi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{g}{N}\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5 \psi)^2\right]$$

Respectively $O(4N)$ and $U(N) \times U(1)$ symmetry.

Non-trivial CFT in the ultraviolet (UV).

Convenient to introduce collective fields

$$\sigma = \frac{g}{N}\bar{\psi}\psi$$

$$\varphi = \frac{g}{N}\left(\bar{\psi}\psi + \bar{\psi}\gamma^5 \psi\right)$$

GNY AND NJL MODELS AT LARGE N

$$S_{GNY}[\psi,\sigma] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \left(\gamma^\mu \partial_\mu + \sigma\right) \psi_i + \frac{1}{2g_Y} \partial_\mu \sigma \partial_\mu \sigma.$$

$$S_{NJL}[\psi,\varphi] = \int dt\, d\Sigma \sum_{i=1}^{N} \bar{\psi}_i \left[\gamma^\mu \partial_\mu + \varphi \left(\frac{1+\gamma_5}{2}\right) + \varphi^* \left(\frac{1-\gamma_5}{2}\right)\right] \psi_i + \frac{1}{g_Y} \partial_\mu \varphi^* \partial_\mu \varphi$$

They flow in the IR to the same CFTs.

We want to fix $U(1)$ charges.

$$\psi_i \to e^{i\alpha} \psi_i,$$

$$\psi \to e^{i\alpha\gamma_5} \psi, \qquad \varphi \to e^{-2i\alpha} \varphi,$$

and compute the partition function at fixed charge

$$Z(Q) = \mathrm{Tr}\left[e^{-\beta H} \delta(\hat{Q} - Q)\right].$$

FIX THE CHARGE

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta Q}\, \mathrm{Tr}\left[e^{-\beta H} e^{-i\theta \hat{Q}}\right].$$

At large N the integral over θ becomes a Legendre transform

$$\Delta(Q) = -\frac{1}{\beta}\log\left(Z_{S^2}(Q)\right) = \sup_{i\theta}\left(i\theta Q - S_{\text{eff}}(\theta)\right)$$

The trace is written as a path integral in two ways:

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\substack{\psi(2\pi\beta)=-e^{-\theta}\psi(0) \\ \varphi(2\pi\beta)=e^{2i\theta}\varphi(0)}} \mathcal{D}\varphi_i\, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\substack{\psi(2\pi\beta)=-\psi(0) \\ \varphi(2\pi\beta)=\varphi(0)}} \mathcal{D}\varphi_i\, e^{-S^\theta[\varphi]}$$

EFFECTIVE ACTION: COVARIANT DERIVATIVE

The actions are quadratic in the fermions. We can integrate them out. For the bosonic fields, we expand as vacuum expectation value (VEV) plus fluctuations

$$\sigma = \sigma_0 + \frac{1}{\sqrt{N}}\hat{\sigma}$$

$$\varphi_0 = \varphi_0 + \frac{1}{\sqrt{N}}\hat{\varphi}$$

The leading contribution comes from the VEV:

$$\Omega = S_{eff} = -N\,\mathrm{Tr}\log\left(\gamma^\mu\partial_\mu - i\frac{\theta}{\beta}\gamma_3 + \sigma_0\right)$$

$$\Omega = S_{eff} = -N\,\mathrm{Tr}\log\left(\gamma^\mu\partial_\mu - i\frac{\theta}{\beta}\gamma_3\gamma_5 + \varphi_0\left(\frac{1+\gamma_5}{2}\right) + \varphi_0^*\left(\frac{1-\gamma_5}{2}\right)\right)$$

This has to be minimized with respect to σ_0 and φ_0 (gap equation).

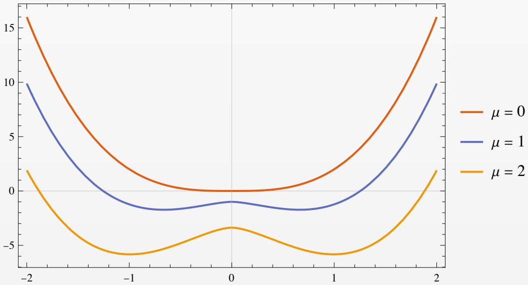
We can read $i\theta/\beta = \mu$ as a chemical potential.

THE NAMBU–JONA–LASINIO MODEL

In the $\beta \to \infty$ limit, on the torus $\Sigma = T^2$, the grand potential is

$$\frac{\Omega}{N} = -\int \frac{d^2p}{(2\pi)^2}\left[\sqrt{(|p|+\mu)^2+|\varphi_0|^2}+\sqrt{(|p|-\mu)^2+|\varphi_0|^2}\right]$$

$$= -\frac{1}{6\pi}\left[3|\varphi|^2\mu\,\mathrm{arctanh}\,\frac{\mu}{\sqrt{|\varphi|^2+\mu^2}}+(\mu^2-2|\varphi|^2)\sqrt{|\varphi|^2+\mu^2}\right]$$



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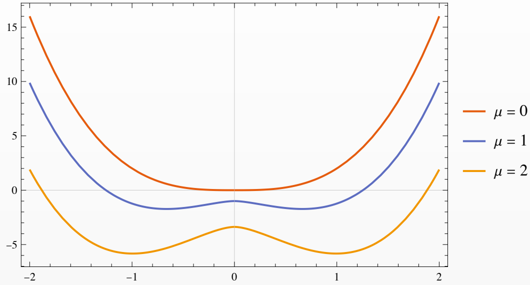


The gap equation admits a non-vanishing solution for any value of μ

$$\varphi_0 = \mu\sqrt{\kappa_0^2 - 1} = 0.6627\cdots\times\mu$$

This is the physics that we had discussed before: fixing the charge induces a spontaneous symmetry breaking.

The field φ is the order parameter for the superfluid phase transition.

COOPER PAIRS

The scalar φ can be understood as a composite field

$$\varphi = \bar{\psi}\psi + \bar{\psi}\gamma^5\psi$$

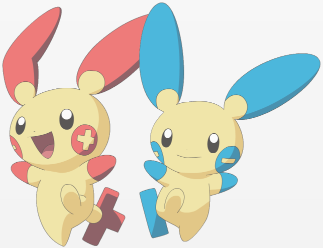
Its meaning is even more transparent after a Pauli–Gürsey transformation:

$$\psi \mapsto \frac{1}{2}\left[(1-\gamma_5)\psi - (1+\gamma_5)C\bar{\psi}^T\right], \qquad \bar{\psi} \mapsto \frac{1}{2}\left[\bar{\psi}(1+\gamma_5) - \psi^T C(1-\gamma_5)\right].$$

because then we identify φ with a Cooper pair

$$\varphi = \psi^t C\psi$$

In presence of an attractive interactions, fermion form pairs that behave as bosons and undergo a Bose-Einstein transition.



CONFORMAL DIMENSIONS

Now we can repeat the computation on S^2.

By the state-operator correspondence, the free energy is the conformal dimension of the lowest operator.

$$\Omega = -\frac{N}{4\pi r_0^2} \sum_{j=1/2}^{\infty} (2j+1)\left(\Omega_+ + \Omega_-\right), \qquad \Omega_\pm = |\varphi|^2 + (\omega_j \pm \mu)^2,$$

where $\omega_j = j + 1/2$ are the eigenvalues on the sphere.

We need to minimize with respect to φ and Legendre transform from the chemical potential μ to the charge Q.

Two regimes:

- The large-charge regime $Q \gg N$,
- The small-charge regime $Q \ll N$

LARGE CHARGE

We need to regularize the sums (ask). The grand potential has two pieces:

$$\Omega_r = -\frac{2N}{3}(r_0\mu)^3\left(3(\kappa_0^2-1)\operatorname{arccoth}\kappa_0 + 3\kappa_0 - 2\kappa_0^3 + 2(\kappa_0^2-1)^{3/2}\right) + \dots,$$

$$\Omega_d = N\left(\frac{4(\varphi_0 r_0)^3}{3} + \frac{\varphi_0 r_0}{3} - \frac{1}{60\varphi_0 r_0} + \dots\right).$$

In the large-charge regime we look for a solution of the form

$$\varphi_0 r_0 = \sqrt{\kappa_0^2 - 1}\left(\mu r_0 + \frac{\kappa_1}{\mu r_0} + \frac{\kappa_2}{(\mu r_0)^3} + \dots\right).$$

After some algebra we can solve the gap equation order-by-order

$$\kappa_0 \tanh\kappa_0 = 1, \qquad \kappa_1 = -\frac{1}{12\kappa_0^2}, \qquad \kappa_2 = \frac{33-16\kappa_0^2}{1440\kappa_0^6}, \qquad \dots$$

ORDER N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N\kappa_0}\right)^{1/2} - \frac{11-6\kappa_0^2}{360\kappa_0^2}\left(\frac{Q}{2N\kappa_0}\right)^{-1/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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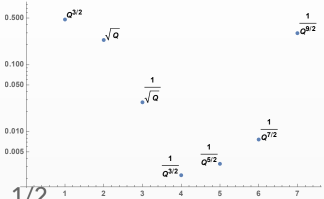

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SMALL CHARGE

In the small charge regime, we need again to separate the regular from the divergent part:

$$
\begin{aligned}
\Omega_r &= -2N \sum_{\ell=1} \ell \Bigg[\sqrt{(\ell + \tfrac{1}{2} + \hat{\mu} r_0)^2 + (\varphi_0 r_0)^2} + \sqrt{(\ell - \tfrac{1}{2} - \hat{\mu} r_0)^2 + (\varphi_0 r_0)^2} \\
&\qquad - \sqrt{(\ell + \tfrac{1}{2})^2 + (\varphi_0 r_0)^2} + \sqrt{(\ell - \tfrac{1}{2})^2 + (\varphi_0 r_0)^2} \Bigg], \\
\Omega_d &= -4N \sum_{\ell=1}^{\infty} (\ell + \tfrac{1}{2}) \sqrt{(\ell + \tfrac{1}{2})^2 + (\varphi_0 r_0)^2}.
\end{aligned}
$$

The divergent part can be understood in terms of zeta functions (ask)

SMALL CHARGE

Now we expand around $\mu = 1/(2r_0)$ in powers of μ.
This is the conformal coupling to a cylinder in three dimensions.

$$\mu = 1/(2r_0) + \mu_2 \varphi_0^2 r_0 + \mu_4 \varphi_0^4 r_0^3 + \ldots.$$

With some algebra

$$\frac{\Delta(Q)}{2N} = \frac{1}{2}\left(\frac{Q}{2N}\right) + \frac{2}{\pi^2}\left(\frac{Q}{2N}\right)^2 + \ldots$$

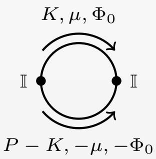
Consistent with the fact that φ has charge two and dimension 1/2, so that $\Delta(Q) \approx Q/2$.

ORDER N°

In the EFT we had a universal term. Is it really universal?
Here it can only appear at order 1/N, so we need to compute the fluctuations around the vacuum. The fermion propagator is

$$D_\psi(P) = (-i\not{P} + \varphi_0 - \mu\Gamma_3\Gamma_5)^{-1}.$$

The fluctuations for the scalars are obtained with a one-loop fermion computation. For example the real part of φ interacts with itself via



$$= D^{-1}(P) = -\int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\left[D_\psi(K) D_\psi(P-K)\right],$$

ORDER N°: THE UNIVERSAL GOLDSTONE

The final result for the two real components is

$$D^{-1}(P) = \begin{pmatrix} \frac{\kappa_0 \mu}{\pi} + \frac{2\kappa_0^2\left(2\kappa_0^2-1\right)\omega^2+\left(3\kappa_0^6-2\kappa_0^4-2\kappa_0^2+2\right)p^2}{24\pi\kappa_0^3(\kappa_0^2-1)\mu} & -\frac{\kappa_0}{2\pi}\omega \\ \frac{\kappa_0}{2\pi}\omega & \frac{2\kappa_0\omega^2+\kappa_0^3 p^2}{8\pi(\kappa_0^2-1)\mu} \end{pmatrix} + \mathcal{O}(P^3/\mu^3),$$

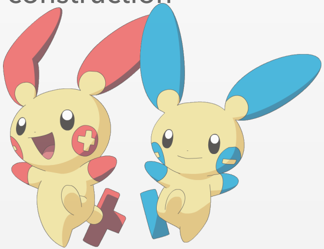
and from here we read the dispersion relations for a massive and for the universal Goldstone mode

$$\omega_1^2 = \frac{1}{2}p^2 + \ldots,$$

$$\omega_2^2 = 12\kappa_0^4\frac{\kappa_0^2-1}{2\kappa_0^2-1}\mu^2 + \frac{5\kappa_0^6-5\kappa_0^4-\kappa_0^2+2}{2\kappa_0^2(2\kappa_0^2-1)}p^2 + \ldots$$

WHAT HAS HAPPENED?

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge results in **spontaneous symmetry breaking**
- The order parameter is a composite field (**Cooper pair**)
- We compute the conformal dimension of the lowest operator of charge Q
- We find a result in **total agreement** with the general EFT construction



THE GROSS–NEVEU–YUKAWA MODEL

Now for the Gross-Neveu-Yukawa model.
Integrating out the Matsubara frequencies we find

$$\frac{\Omega}{N} = -2 \int \frac{d^2 p}{(2\pi)^2} \left\{ \omega_p + \frac{1}{\beta} \log\left(1 + e^{-\beta(\omega_p + \mu)}\right) + (\mu \leftrightarrow -\mu) \right\},$$

with $\omega_p^2 = p^2 + \sigma_0^2$. The gap equation is

$$\sigma_0 - \frac{1}{\beta} \log\left((1 + e^{\beta(\sigma_0 + \mu)})(1 + e^{\beta(\sigma_0 - \mu)}) \right) = 0$$

and admits only the solution $\sigma_0 = 0$.
In other words, in the large-N limit the symmetry is never broken.
This is different from the superfluid EFT behavior.

THE GROSS–NEVEU–YUKAWA MODEL

We can repeat the computation on the sphere. The grand potential is

$$\frac{\Omega}{N} = -\frac{1}{2\pi r_0^2}\left[\sum_{\omega_j>\mu}(2j+1)\omega_j + \mu\sum_{\omega_j<\mu}(2j+1)\right]$$

the solution to the gap equation and the Legendre transform are

$$\frac{Q}{N} = \frac{1}{2\pi r_0^2}\lfloor \mu r_0\rfloor(\lfloor \mu r_0\rfloor + 1), \qquad \frac{E}{N} = \frac{1}{6\pi r_0^3}\lfloor \mu r_0\rfloor(\lfloor \mu r_0\rfloor + 1)(2\lfloor \mu r_0\rfloor + 1).$$

This is the physics of a Fermi sphere.

However, the behavior of the conformal dimension is still the same

$$\Delta = \frac{2}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{1}{12}\left(\frac{Q}{2N}\right)^{1/2} - \frac{1}{192}\left(\frac{Q}{2N}\right)^{-1/2} + \ldots$$

WHAT HAS HAPPENED?

- We have taken a model in which the fixed point is under large-N control
- Fixing the charge **does not result** in spontaneous symmetry breaking
- The physics is the one of a **Fermi sphere**
- The dimension of the lowest operator still obeys the same law
- **Conundrum:** is this a large-N effect or is there a finite-N transition?

RESURGENCE AND THE LARGE CHARGE

RESULTS FROM LARGE N

O(2N) at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.

$$\begin{cases} F_{\Sigma}(Q) = \mu Q + N\zeta(-\frac{1}{2}|\Sigma, \mu), \\ \mu\zeta(\frac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}. \end{cases}$$

RESULTS FROM LARGE N

O(2N) at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.
The free energy per DOF $f(\hat{q}) = F/(2N)$ is

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)), \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

RESULTS FROM LARGE N

O(2N) at criticality in $1+2$ dimensions on $\mathbb{R}\times\Sigma$. Double-scaling limit $N\to\infty$, $Q\to\infty$ with $\hat{q}=Q/(2N)$ fixed.
The free energy per DOF $f(\hat{q})=F/(2N)$ is

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$\zeta(s|\Sigma,\mu)$ is the zeta function for the operator $-\triangle+\mu^2$. In Mellin representation

$$\zeta(s|\Sigma,\mu) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t}\,\mathrm{Tr}\left(e^{\triangle t}\right).$$

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Large $\hat{q}$ is large μ and is small t. The classical Seeley-de Witt problem:

$$\operatorname{Tr}\left(e^{\triangle t}\right) \sim \frac{V}{4\pi t}\left(1 + \frac{R}{12} t + \dots\right).$$

THE TORUS

As a warm-up: $\Sigma = T^2$.

$$\mathrm{spec}(\triangle) = \left\{ -\frac{4\pi^2}{L^2}\left(k_1^2 + k_2^2\right) \mid k_1, k_2 \in \mathbb{Z} \right\}.$$

It follows that the heat kernel trace is the square of a theta function:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}(k_1^2 + k_2^2)t} = \left[\theta_3\left(0, e^{-\frac{4\pi^2 t}{L^2}}\right)\right]^2.$$

We are interested in the small-t limit: we Poisson-resum the series:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1 + \sum_{k \in \mathbb{Z}} e^{-\frac{k^2 L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{\|\mathbf{k}\|^2 L^2}{4t}}\right)$$

THE TORUS

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|T^2,\mu) = \frac{L^2\mu^3}{12\pi}\left(1 + \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\mu L}}{\|\mathbf{k}\|^2\mu^2 L^2}\left(1 + \frac{1}{\|\mathbf{k}\|\mu L}\right)\right).$$

Free energy

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}\left(1 - \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\sqrt{4\pi\hat{q}}}}{8\|\mathbf{k}\|^2\pi\hat{q}} + \dots\right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/\hat{q}$ and $e^{-\sqrt{4\pi\hat{q}}}$.
- non-perturbative effects more important than the “usual” instantons $\mathcal{O}\left(e^{-\hat{q}}\right)$

THE SPHERE

On the two sphere $\mathrm{spec}(\triangle) = \{-\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell + 1$.

Again, we use Poisson resummation

$$\mathrm{Tr}\left(e^{\triangle t}\right)e^{-t/4} = \sum_{\ell\geq 0}(2\ell+1)e^{-(\ell+1/2)^2 t} \sim \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(1-2^{1-2n})}{n!}B_{2n}t^n$$

The series is asymptotic: the Seeley-de Witt coefficients diverge like n!:

$$a_n = \frac{(-1)^{n+1}(1-2^{1-2n})}{n!}B_{2n} \sim \frac{2n^{1/2}}{\pi^{5/2+2n}}n!.$$

this divergence is reflected in the existence of non-perturbative corrections.

BOREL RESUMMATION

BOREL TRANSFORM

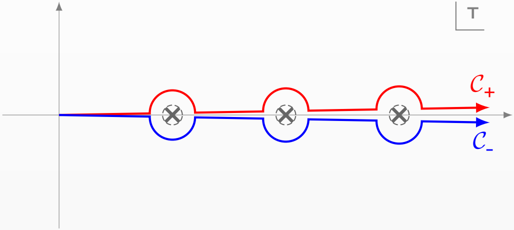
We need to make sense of the divergent series and the imaginary terms.

$$H(t) = \sum_{n \geq 0} a_n t^n \longrightarrow \hat{H}(\tau) = \sum_{n \geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

LATERAL TRANSFORM

If there are poles on the real positive axis there is an ambiguity



$$s_{\pm}(H)(t) = s(H)(t) = \int_{\mathcal{C}_{\pm}} w^b e^{-w} \hat{H}(tw^{\beta}) \frac{dw}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.

MORE INGREDIENTS

WORLDLINE INTERPRETATION

We need a **non-perturbative interpretation** of these exponential terms.

We read the heat kernel as the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \triangle$, i.e. a **free quantum particle moving on $\mathbb{R} \times \Sigma$**.

We can write the partition function as a **path integral**

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X \, e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t} \int_0^1 d\tau \, g_{\mu\nu} \dot{X}^\mu(\tau) \dot{X}^\nu(\tau)$$

A TRANSSERIES FROM GEODESICS

In the limit $t \to 0$ the path integral localizes on a sum over all the closed geodesics γ.

For each geodesic a perturbative series in t, weighted by $e^{-\ell(\gamma)^2/(4t)}$

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

$$= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma \in \text{closed geodesics}} e^{-\frac{\ell(\gamma)^2}{4t}} t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,$$

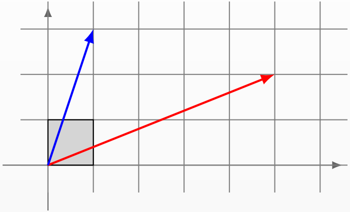
the b_γ depend on the geometry.

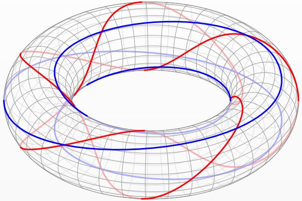
This is precisely the same structure predicted by resurgence.

Now we have a geometric interpretation.

THE TORUS

In the case of the torus, closed geodesics are labelled by two integers (k_1, k_2)





The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h \, e^{-\frac{1}{4t}\int_0^1 d\tau (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t}\partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$

THE TORUS

Now we can write the result of the path integral

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X \, e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{cl}} \int_{h(1)=h(0)=0} e^{-S[X_{cl}]-S[h]}$$

$$= \mathcal{N} L^2 \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 (k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h \, e^{-S[h]},$$

$$= \frac{L^2}{4\pi t}\left[1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 \|k\|^2}{4t}}\right]$$

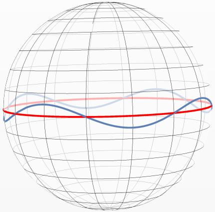
This is exactly what we had found before just by looking at the spectrum. Now we can understand the non-perturbative effects in terms of closed geodesics.

THE SPHERE

Closed geodesics on the sphere go around the equator k times

THE SPHERE

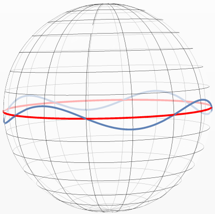
Closed geodesics on the sphere go around the equator k times

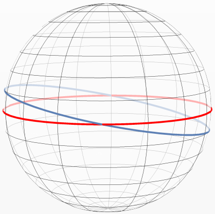


We need to sum over the fluctuations h_φ and h_θ

THE SPHERE

Closed geodesics on the sphere go around the equator k times





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There is a zero mode because we can rotate the equator

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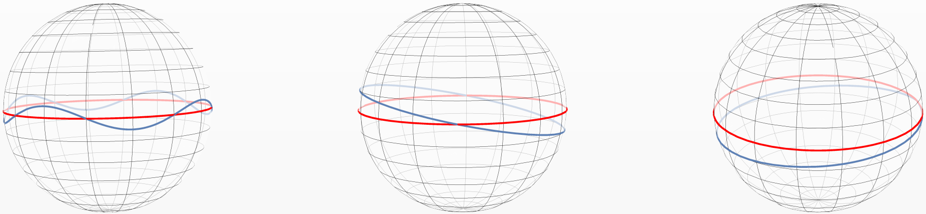

We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator

And an instability because we can slide off

BACK TO RESURGENCE

Putting it **all together**, the non-trivial geodesics give

$$\pm 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\in\mathbb{Z}}|k|e^{-\frac{k^2\pi^2}{t}}$$

The one-loop result **perfectly cancels** the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr}\left(e^{(\Delta-\frac{1}{4})t}\right) = s_{\pm}(H)(t) \mp 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\geq 1}(-1)^k k e^{-\frac{k^2\pi^2}{t}} = \mathrm{Re}[s_{\pm}(H)(t)]$$

BACK TO RESURGENCE

We can write the exact expression for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \mathrm{Re}\left[\frac{2rm^2}{\pi}\int_0^\infty dy\,\frac{K_2(2mry)}{y\sin(y)}\right] = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

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As a numerical test, we can compare with the convergent small-charge expansion ($\hat{q} \approx 0.6$)

$$r\omega(mr = 0.4)\Big|_{\text{small charge}} = 0.012\,777\,296\,63\ldots$$

$$r\omega(mr = 0.4)\Big|_{\text{resurgence}} = 0.012\,777\,297\,69\ldots$$

OPTIMAL TRUNCATION

LESSONS FROM LARGE N

Let's go back to the EFT.
The effective action is identified with the asymptotic expansion: the **grand potential** is the value of the **action at the minimum** $\chi = \mu t$:

$$\omega(\mu) = L_{EFT}\Big|_{\chi=\mu t}$$

where

$$L_{EFT} = \omega_0 \left(\partial_\mu \chi \partial^\mu \chi\right)^{3/2} + \omega_1 \left(\partial_\mu \chi \partial^\mu \chi\right)^{1/2} + \dots,$$

In general the **coefficients are unknown**

BUT

Now we have a **geometric understanding** of the non-perturbative effects

LESSONS FROM LARGE N

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is
   a particle of mass μ going around the equator.

A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n\geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi\kappa f_0^{(0)} \sqrt{Q}} \sum_{n\geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \ldots$$

(we used $\mu = 3f_0^{(0)}\sqrt{Q}/2 + \ldots$)

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(we used $\mu = 3 f_0^{(0)} \sqrt{Q}/2 + \dots$)

LESSONS FROM LARGE N

- The **controlling parameter** for the non-perturbative effects $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ is fixed by the **leading term** in the $1/Q$ expansion.
- The non-perturbative coefficient $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ fixes the **large-n behavior** of the perturbative series $f_n^{(0)}$.

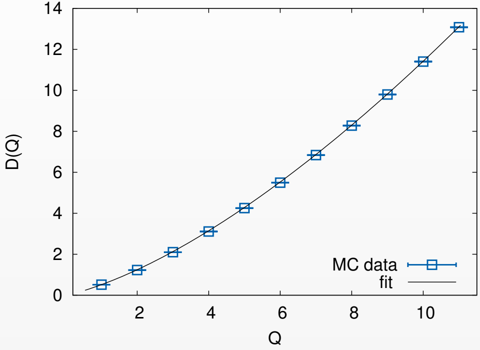
$$f_n^{(0)} \sim (2n)!(3\pi\kappa f_0^{(0)})^{-n}$$

We don't know enough for a Borel resummation, but we can estimate an optimal trucation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi\kappa f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$

CAN WE UNDERSTAND THE LATTICE RESULTS NOW?



In $O(2)$, $f_0^{(0)} \approx 0.301(3)$
so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\pi\sqrt{Q}}\right)$.

Lattice: Best fit with $N = 3$ terms.
At $Q = 1$ the error is $\approx 6 \times 10^{-2}$; at $Q = 11$ the error is $\approx 5 \times 10^{-5}$.

Resurgence: $\sqrt{10} \approx 3.16$
$e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$.

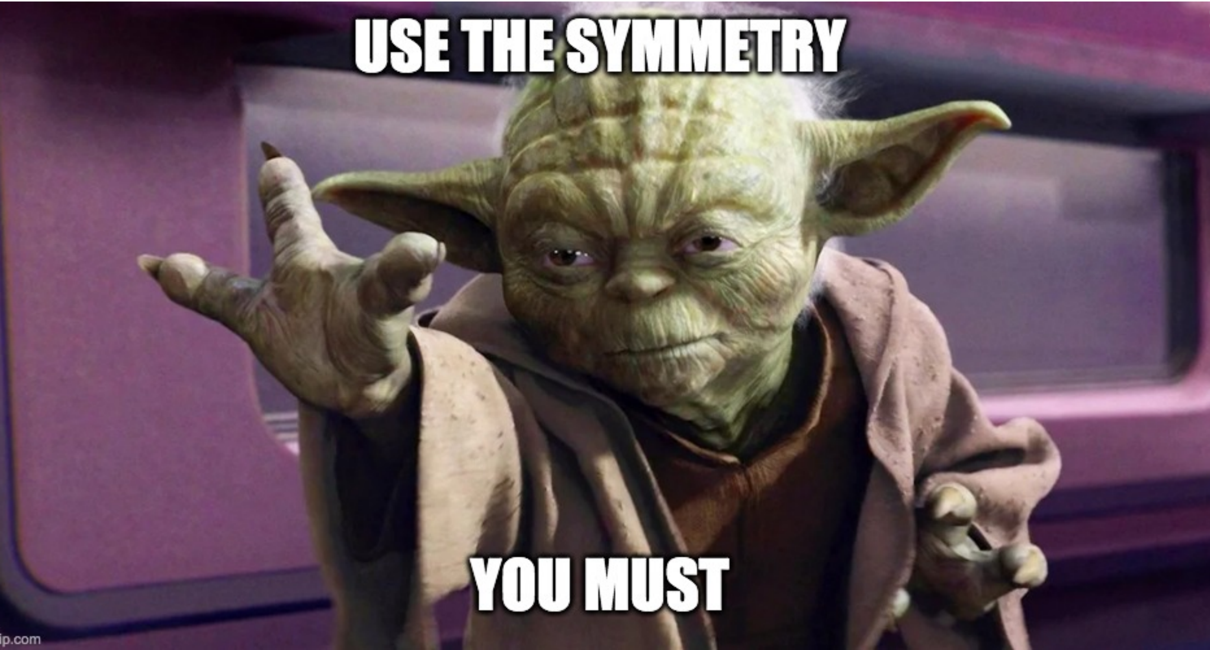
WHAT HAS HAPPENED?

- The large-charge expansion of the Wilson-Fisher point is **asymptotic**
- In the **double-scaling** limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion
- We can **Borel**-resum the expansion
- We have a **geometric interpretation for the non-perturbative effects**
- We can use this geometric interpretation also in the **finite-N** case
- We obtain an **optimal truncation** and estimate of the error
- The results are **consistent with lattice simulations**

CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Precise and **testable predictions**.
- Qual(nt)itative control of the **non-pertubative** effects.
- **CFT constraints**: perturbative/non-perturbative **interplay**.
- Remarkable agreement with **lattice**.

AN EFT FOR A CFT



THE O(2) MODEL

The simplest example is the WF point of the O(2) model in three dimensions.

- Non-trivial fixed point of the φ^4 action

$$L_{UV} = \partial_\mu \varphi^* \partial_\mu \varphi - u(\varphi^* \varphi)^2$$

- Strongly coupled
- In nature: ^{4}He.
- Simplest example of spontaneous symmetry breaking.
- **Not accessible** in perturbation theory. **Not accessible** in 4 – ε. **Not accessible** in large N.
- Lattice. Bootstrap.

CHARGE FIXING

We consider a **subsector of fixed charge Q**.

Generically, the classical solution at fixed charge **breaks spontaneously $U(1) \to \emptyset$**.

We have one **Goldstone boson** χ.

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2}\partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

We want to describe a CFT: we can **dress with a dilaton**

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2}\partial_\mu \chi \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2}\left(\partial_\mu \sigma \partial_\mu \sigma - \frac{\xi R}{f^2}\right)$$

The fluctuations of χ give the Goldstone for the broken U(1), the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

LINEAR SIGMA MODEL

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.

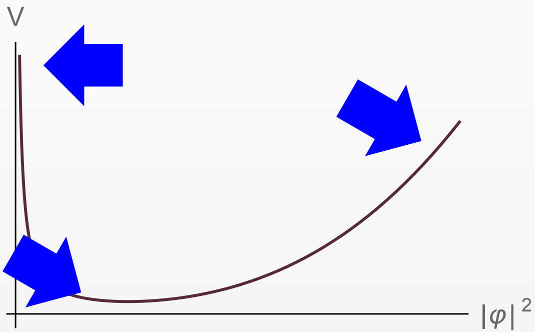
Scale invariance is manifest.

The field φ is some complicated function of the original φ.

CENTRIFUGAL BARRIER

The O(2) symmetry acts as a shift on χ.

Fixing the charge is the same as adding a **centrifugal term** $\propto \frac{1}{|\varphi|^2}$.



GROUND STATE

We can find a fixed-charge solution of the type

$$\chi(t, x) = \mu t \qquad \sigma(t, x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \ldots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2}/\sqrt{V}Q^{3/2} + c_{1/2}R\sqrt{V}Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

FLUCTUATIONS

The fluctuations over this ground state are described by two modes.

- A universal "**conformal Goldstone**". It comes from the breaking of the U(1).

$$\omega = \frac{1}{\sqrt{2}} p$$

- The **massive dilaton**. It controls the magnitude of the quantum fluctuations. **All quantum effects are controled by 1/Q.**

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

NON-LINEAR SIGMA MODEL

Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.

All other terms are suppressed by powers of 1/Q.

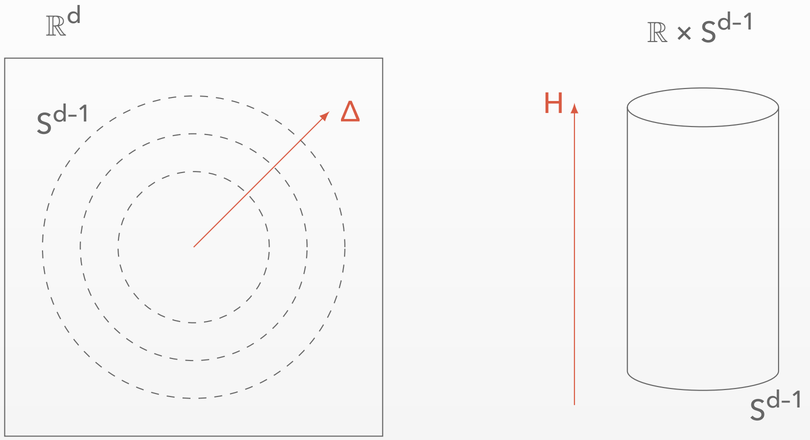
In 3 + 1 NRCFT the analogous story in a background potential A_0 leads to

$$L[\chi] = c_0 U^{5/2} + c_1 U^{-1/2} \partial_i U \partial_i U + c_2 U^{1/2}((\partial_i \partial_i \chi)^2 - 9 \partial_i \partial_i A_0) + \dots \qquad (1)$$

where $U = \partial_t \chi - A_0 \chi - \partial_i \chi \partial_i \chi / 2$.

STATE-OPERATOR CORRESPONDENCE

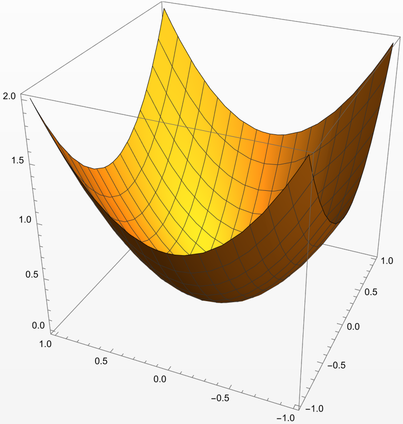
The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.

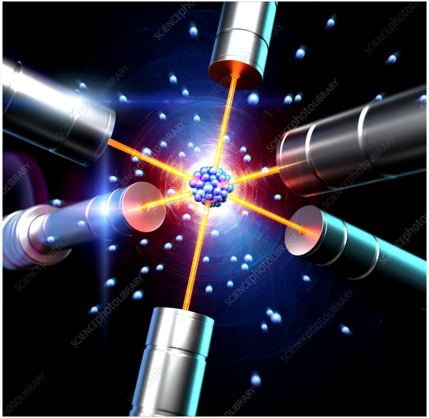


Protected by conformal invariance: a well-defined quantity.

NRCFT STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in a harmonic trap.





Protected by conformal invariance: a well-defined quantity.

CONFORMAL DIMENSIONS

We know the energy of the ground state.

The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}}\zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the **conformal dimension of the lowest operator of charge Q** in the O(2) model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}}Q^{3/2} + 2\sqrt{\pi}c_{1/2}Q^{1/2} - 0.094\ldots + \mathcal{O}\left(Q^{-1/2}\right)$$

In 3 + 1 NRCFT we find

$$\Delta_Q = c_{4/3}Q^{4/3} + c_{2/3}Q^{2/3} + b_{5/9}Q^{5/9} + b_{1/3}Q^{1/3} + b_{1/9}Q^{1/9} - \frac{1}{3\sqrt{3}}\log(Q) + c_0$$

WHAT HAPPENED?

We started from a CFT.
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple EFT**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

ORDER N°

The order N^0 terms are

$$S^{\theta}[\hat{\sigma}, \hat{\lambda}] = \int dt\, d\Sigma \left((D_{\mu}\hat{\sigma})^{*} (D^{\mu}\hat{\sigma}) + (\mu^2 + \hat{\lambda})\hat{\sigma}^{*}\hat{\sigma} + \frac{\hat{\lambda} v(\hat{\sigma} + \hat{\sigma}^{*})}{(N-1)^{1/2}} \right)$$
$$+ \frac{1}{2} \int dx_1\, dx_2\, \hat{\lambda}(x_1)\hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x - y)$ is the propagator $(D_{\mu}D^{\mu} + m^2)^{-1}$.

At low energies we can approximate the non-local term as

$$\int dt\, d\Sigma\, \hat{\lambda}(x)^2 \zeta(2|\theta, \Sigma, \mu) \approx \frac{V}{2\mu} \int dt\, d\Sigma\, \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

ORDER N°

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4\mu^2) & \mu\omega \\ -\mu\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \ldots = 0 \qquad \omega^2 + 8\mu^2 + \frac{3}{2}p^2 + \ldots = 0$$

This is the conformal Goldstone that we have seen in the EFT.

Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\ldots$$

This is **universal**. Does not depend on N or Q.

HIGHER ORDERS

There are infinite non-local terms

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx_1 \ldots dx_n \, \hat{\lambda}(x_1) \ldots \hat{\lambda}(x_n) P(x_1, \ldots, x_n)$$

At low energy they are approximated by

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx \, \hat{\lambda}(x)^n C_n$$

HIGHER ORDERS

There is only one scale, the charge density ρ = Q/V. We must have

$$C_n = \rho^{3/2-n} c_n$$

So

$$S_{nl} = Q^{3/2} \sum_{n=3}^{\infty} \frac{c_n}{n(N-1)^{n/2-1}} \int dx\, \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by 1/N.

⏩ back to the talk