Large Charge at large N

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Envisaging Future Trajectories in Effective Field Theory | Kashiwa no ha | 15 March 2024

arXiv:1505.01537, arXiv:1610.04495, arXiv:2008.03308, arXiv:2102.12488, arXiv:2110.07616, arXiv:2110.07617 . . .

WHO'S WHO

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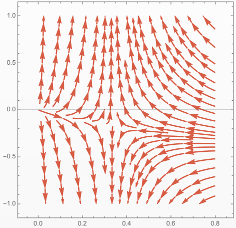
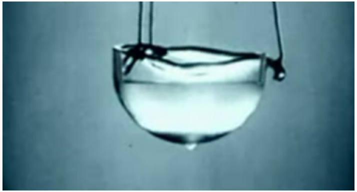
WHY ARE WE HERE? CONFORMAL FIELD THEORIES

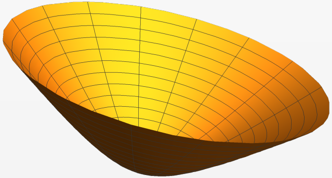
extrema of the RG flow



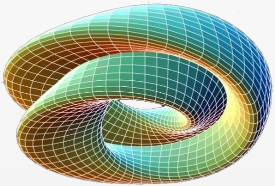
critical phenomena



quantum gravity



string theory



WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

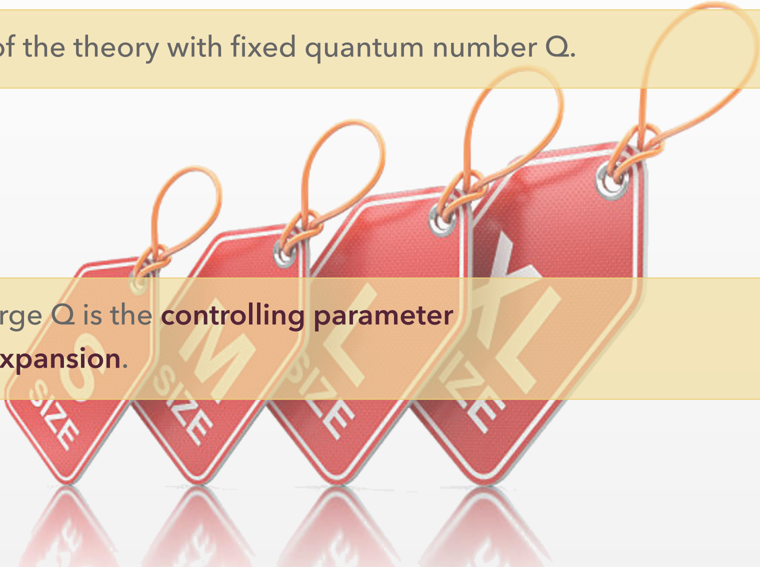
WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

In presence of a **symmetry** there can be **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

THE IDEA

Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.



NOT AN ORIGINAL IDEA

NO BOOTSTRAP HERE!

This approach is **orthogonal to bootstrap**.

We will use an effective action.

We will access sectors that are difficult to reach with bootstrap.

(However, arXiv:1710.11161).

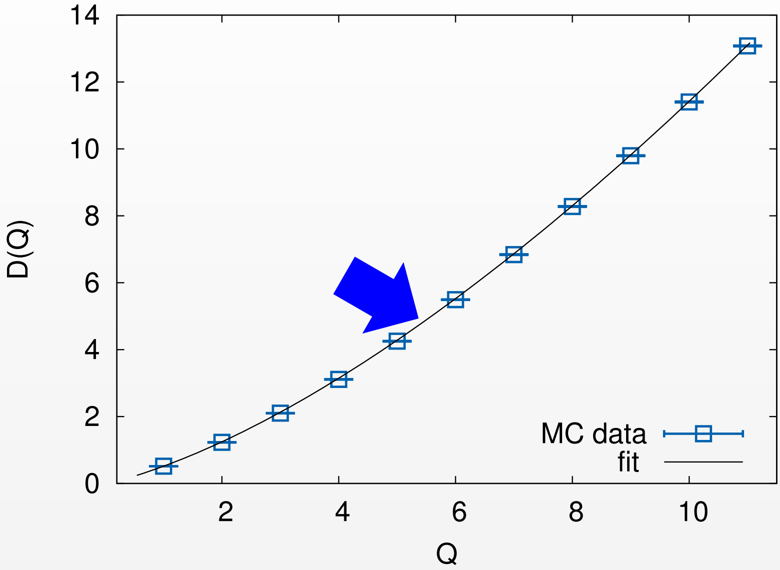
CONCLUSIONS

We consider the O(N) **vector model in three dimensions**. In the IR it flows to a **conformal fixed point** [Wilson & Fisher].

We find an explicit formula for the **dimension of the lowest primary at fixed charge**:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

CONCLUSIONS: O(2)



SCALES

We want to write a **Wilsonian effective action.**

Choose a cutoff Λ, separate the fields into high and low frequency φ_H, φ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\varphi_L)} = \int \mathcal{D}\varphi_H \, e^{iS(\varphi_H,\varphi_L)}$$

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too hard

SCALES

- We look at a finite box of typical **length R**
- The U(1) charge Q fixes a **second scale $\rho^{1/2} \sim Q^{1/2}/R$**

$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the **effective action is weakly coupled and under perturbative control** in powers of ρ^{-1}.

NON-LINEAR SIGMA MODEL

In a generic theory™, picking the lowest state of fixed charge induces a spontaneous symmetry breaking.
The low-energy physics is described by a **Goldstone field** χ.

Using conformal invariance, the most general action must take the form

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$. All other terms are suppressed by powers of $1/Q$.

NON-LINEAR SIGMA MODEL

The **energy of the lowest state** for this action has the form

$$E = \frac{c_{3/2}}{\sqrt{V}} Q^{3/2} + R\sqrt{V} Q^{1/2} + \ldots$$

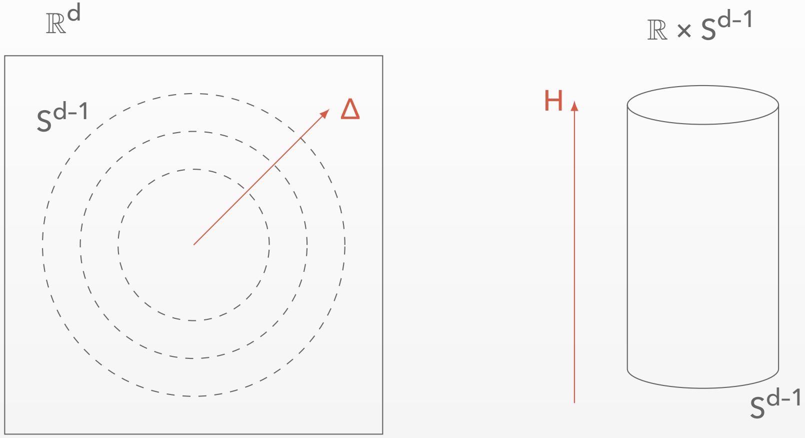
The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}} \zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

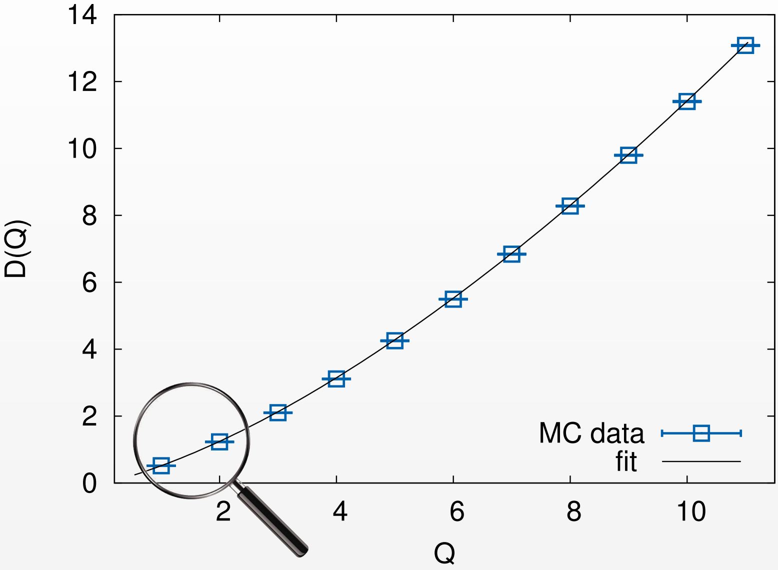
STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

TOO GOOD TO BE TRUE?



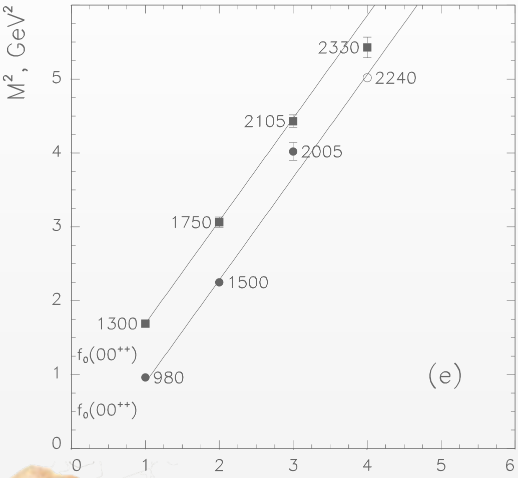
TOO GOOD TO BE TRUE?

Think of **Regge trajectories**.

The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but experimentally everything works so well at small J that String Theory was invented.



CONCLUSIONS: LESSONS FROM RESURGENCE

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is
   a particle of mass μ going around the equator.

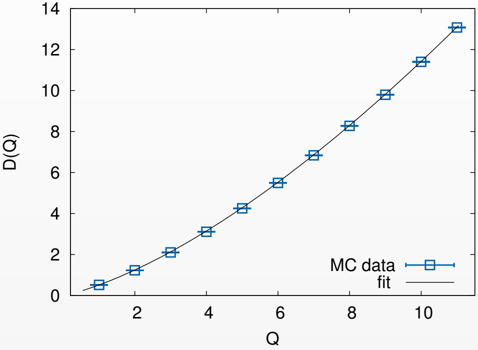
A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n\geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}} \sum_{n\geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \ldots$$

(we used $\mu = 3f_0^{(0)}\sqrt{Q}/2 + \ldots.$)

CAN WE UNDERSTAND THE LATTICE RESULTS NOW?



In O(2), $f_0^{(0)} \approx 0.301(3)$

so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\pi\sqrt{Q}}\right)$.

Lattice: Best fit with N = 3 terms.

At Q = 1 the error is $\approx 6 \times 10^{-2}$; at Q = 11 the error is $\approx 5 \times 10^{-5}$.

Resurgence: $\sqrt{10} \approx 3.16$

$e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$.

WHAT HAPPENED?

We started from a CFT.
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

SELECTED TOPICS IN THE LARGE CHARGE EXPANSION

- **O(2) model** [Hellerman, DO, Reffert, Watanabe] [Monin, Pirtskhalava, Rattazzi, Seibold]
- **fermions** [Komargodski, Mezei, Pal, Raviv-Moshe] [Antipin, Bersini, Panopoulos] [Hellerman, Dondi, Kalogerakis, Moser, DO, Reffert]
- **holography** [Nakayama] [Loukas, DO, Reffert, Sarkar] [de la Fuente] [Guo, Liu, Lu, Pang] [Giombi, Komatsu, Offertaler]
- **large N** [Álvarez-Gaumé, DO, Reffert] [Giombi, Hyman]
- **ε double-scaling** [Badel, Cuomo, Monin, Rattazzi] [Arias-Tamargo, Rodriguez-Gomez, Russo] [Antipin, Bersini, Sannino, Wang, Zhang] [Jack, Jones]
- **non-relativistic CFTs** [Kravec, Pal] [Helllerman, Swanson] [Favrod, DO, Reffert] [DO, Reffert, Pellizzani] [Hellerman, DO, Reffert, Pellizzani, Swanson]
- $\mathcal{N} = 2$ [Hellerman, Maeda] [Hellerman, Maeda, DO, Reffert, Watanabe] [Bourget, Rodriguez-Gomez, Russo] [Grassi, Komargodski, Tizzano] [Cremonesi, Lanza, Martucci]
- **bootstrap** [Jafferis, Zhiboedov]
- **resurgence** [Dondi, Kalogerakis, DO, Reffert] [Antipin, Bersini, Sannino, Torres] [Watanabe]

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would you like to know more?

TODAY'S TALK

The EFT for the O(2) model in 2 + 1 dimensions

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The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

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The EFT for the $O(2)$ model in $2 + 1$ dimensions

Justify and prove all my claims from first principles

Use resurgence for the large-charge EFT

TODAY'S TALK

The EFT for the O(2) model in 2 + 1 dimensions

- An EFT for a CFT.
- The physics at the saddle.
- State/operator correspondence for anomalous dimensions.

Justify and prove all my claims from first principles

Use resurgence for the large-charge EFT

TODAY'S TALK

The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

Use resurgence for the large-charge EFT

TODAY'S TALK

The EFT for the $O(2)$ model in $2 + 1$ dimensions

Justify and prove all my claims from first principles

Use resurgence for the large-charge EFT

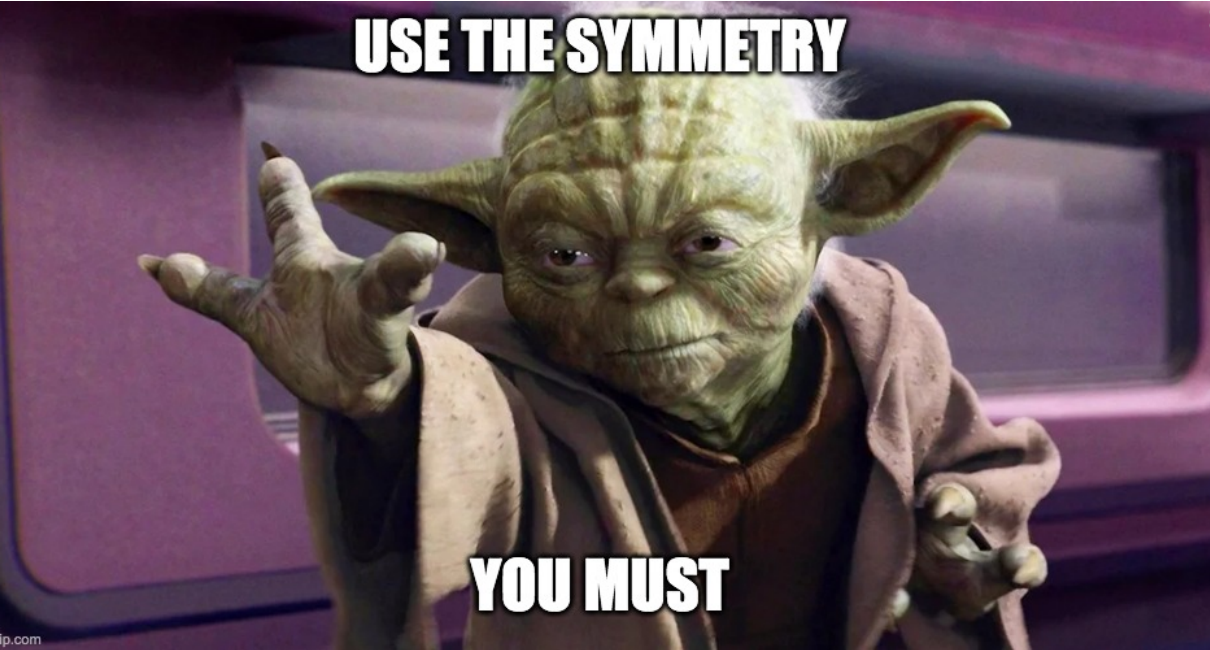
- Borel resum the double-scaling $Q \to \infty$, $N \to \infty$ limit
- geometric interpretation of non-perturbative effects
- general structure of the corrections in the EFT

PARENTAL

ADVISORY

EXPLICIT CONTENT

AN EFT FOR A CFT



THE O(2) MODEL

The simplest example is the Wilson-Fisher (WF) point of the O(2) model in three dimensions.

- Non-trivial fixed point of the φ^4 action

$$L_{UV} = \partial_\mu \varphi^* \partial_\mu \varphi - u(\varphi^* \varphi)^2$$

- Strongly coupled
- In nature: ^4He.
- Simplest example of spontaneous symmetry breaking.
- **Not accessible** in perturbation theory. **Not accessible** in $4 - \varepsilon$. **Not accessible** in large N.
- Lattice. Bootstrap.

CHARGE FIXING

We consider a **subsector of fixed charge Q**.
Generically, the classical solution at fixed charge **breaks spontaneously $U(1) \to \emptyset$**.

We have one **Goldstone boson** χ.

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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We want to describe a CFT: we can **dress with a dilaton**

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken U(1), the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

LINEAR SIGMA MODEL

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.

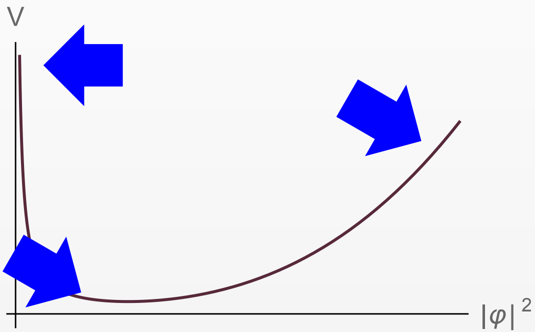
Scale invariance is manifest.

The field φ is some complicated function of the original φ.

CENTRIFUGAL BARRIER

The $O(2)$ symmetry acts as a shift on χ.

Fixing the charge is the same as adding a **centrifugal term** $\propto \dfrac{1}{|\varphi|^2}$.



GROUND STATE

We can find a fixed-charge solution of the type

$$\chi(t,x) = \mu t \qquad \sigma(t,x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \ldots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2}/\sqrt{V}Q^{3/2} + c_{1/2}R\sqrt{V}Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

FLUCTUATIONS

The fluctuations over this ground state are described by two modes.

- A universal "**conformal Goldstone**". It comes from the breaking of the U(1).

$$\omega = \frac{1}{\sqrt{2}} p$$

- The **massive dilaton**. It controls the magnitude of the quantum fluctuations. **All quantum effects are controled by 1/Q.**

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

NON-LINEAR SIGMA MODEL

Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi \, \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \, \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.
All other terms are suppressed by powers of $1/Q$.

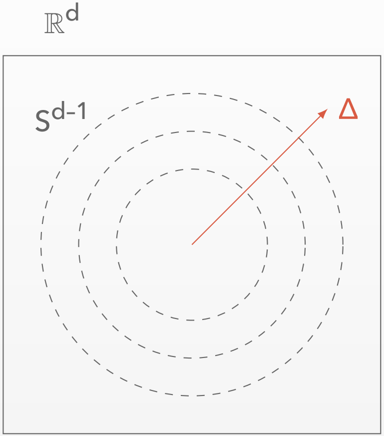
In 3 + 1 NRCFT the analogous story in a background potential A_0 leads to

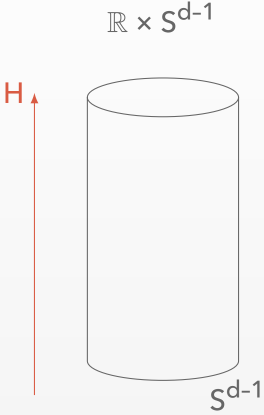
$$L[\chi] = c_0 U^{5/2} + c_1 U^{-1/2} \partial_i U \, \partial_i U + c_2 U^{1/2} \left((\partial_i \partial_i \chi)^2 - 9 \, \partial_i \partial_i A_0 \right) + \dots \qquad (1)$$

where $U = \partial_t \chi - A_0 \chi - \partial_i \chi \, \partial_i \chi / 2$.

STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.

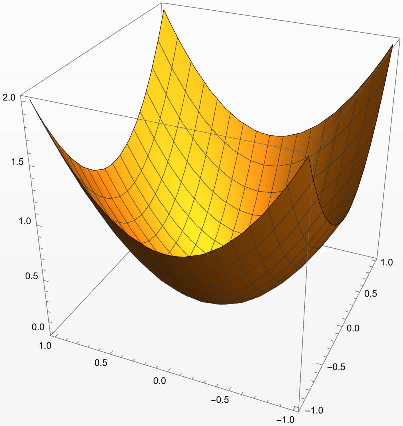


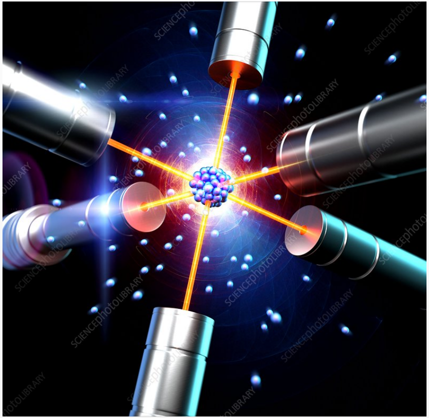


Protected by conformal invariance: a well-defined quantity.

NRCFT STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in a harmonic trap.





Protected by conformal invariance: a well-defined quantity.

CONFORMAL DIMENSIONS

We know the energy of the ground state.

The leading quantum effect is the **Casimir energy of the conformal Goldstone.**

$$E_G = \frac{1}{2\sqrt{2}}\zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the **conformal dimension of the lowest operator of charge Q** in the O(2) model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}}Q^{3/2} + 2\sqrt{\pi}c_{1/2}Q^{1/2} - 0.094\ldots + \mathcal{O}\left(Q^{-1/2}\right)$$

In NRCFT we find

$$\Delta_Q = d_0 Q^{4/3} + d_1 Q^{2/3} + d_2 Q^{5/9} - 0.2941\ldots + \mathcal{O}\left(Q^{-1/2}\right)$$

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LARGE N VS. LARGE CHARGE

THE MODEL

φ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int dt\, d\Sigma \left[g^{\mu\nu} \left(\partial_\mu \varphi_i \right)^* \left(\partial_\nu \varphi_i \right) + r \varphi_i^* \varphi_i + \frac{u}{2} \left(\varphi_i^* \varphi_i \right)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.

We compute the partition function at fixed charge

$$Z(Q_1, \ldots, Q_N) = \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int d\Sigma\, j_i^0 = i \int d\Sigma \left[\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i \right].$$

FIX THE CHARGE

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i} \,\mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i \, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i \, e^{-S^{\theta}[\varphi]}$$

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EFFECTIVE ACTION: COVARIANT DERIVATIVE

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int dt\, d\Sigma \left((D_{\mu}\varphi_i)^* (D^{\mu}\varphi_i) + \frac{R}{8}\varphi_i^* \varphi_i + 2u(\varphi_i^* \varphi_i)^2 \right)$$

$$\begin{cases} D_0 \varphi = \partial_0 \varphi + i\frac{\theta}{\beta}\varphi \\ D_i \varphi = \partial_i \varphi \end{cases}$$

Hubbard–Stratonovich transformation

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int dt\, d\Sigma \left[\left(D^i_{\mu}\varphi_i\right)^* \left(D^i_{\mu}\varphi_i\right) + (r+\lambda)\varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = m^2 + \hat{\lambda}$$

EFFECTIVE ACTION FOR $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_\theta[\hat{\lambda}] = \sum_{i=1}^{N} \left[V\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 + \hat{\lambda} \right) \right] \right].$$

Non-local action for $\hat{\lambda}$.

To be expanded order-by-order in $1/N$.

We can identify the functional determinant with the grand-canonical (fixed chemical potential) free energy:

$$F_{gc}(i\theta) = \sum_{i=1}^{N} \left[V \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \frac{1}{\beta} \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 \right) \right] \right].$$

ZETA FUNCTIONS

In the limit $\beta \to \infty$ (zero temperature), we regularize with a zeta function $\zeta(s|\Sigma, m) = \sum_p (E(p)^2 + m^2)^{-s}$:

The gap equations are (set $A_1 = v$, $A_{>1} = 0$):

$$\frac{\delta}{\delta m} : Vv^2 + \frac{N-1}{2}\zeta(1/2|\Sigma, m) = 0,$$

$$\frac{\delta}{\delta \theta} : -iQ + \frac{2V}{\beta}\theta v^2 = 0,$$

$$\frac{\delta}{\delta v} : 2V\beta\left(m^2 + \frac{\theta^2}{\beta^2}\right)v = 0,$$

For finite Q we need necessarily $v \neq 0$ and then $\theta = im\beta$. So we get

$$m\zeta(1/2|\Sigma, m) = -\frac{Q}{N-1}$$

ORDER N

At leading order in N, the free energy is

$$F(Q) = -\frac{1}{\beta}\left(i\theta Q + N \frac{\partial}{\partial s} \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \beta\zeta(s-1/2|\Sigma, m)\Big|_{s=0} \right)$$

Using the gap equations

$$F(Q) = mQ + N\zeta(-1/2|\Sigma, m)$$

For $\Sigma = S^2$ at large Q/N:

$$F(Q) = \frac{N\sqrt{2}}{3}\left(\frac{Q}{N}\right)^{3/2} + \frac{N}{3\sqrt{2}}\left(\frac{Q}{N}\right)^{1/2} - \frac{7N}{180\sqrt{2}}\left(\frac{Q}{N}\right)^{-1/2} + \dots$$

SMALL Q/N

The zeta function can be expanded in perturbatively in small Q/N.

Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2}\frac{Q}{N} + \frac{16\left(\pi^2 - 12\right)Q^2}{3\pi^4 N^2} + \dots$$

- Expansion of a closed expression
- Start with the engineering dimension 1/2
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation

ORDER N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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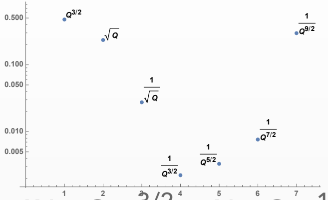

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ORDER N IN NRCFT

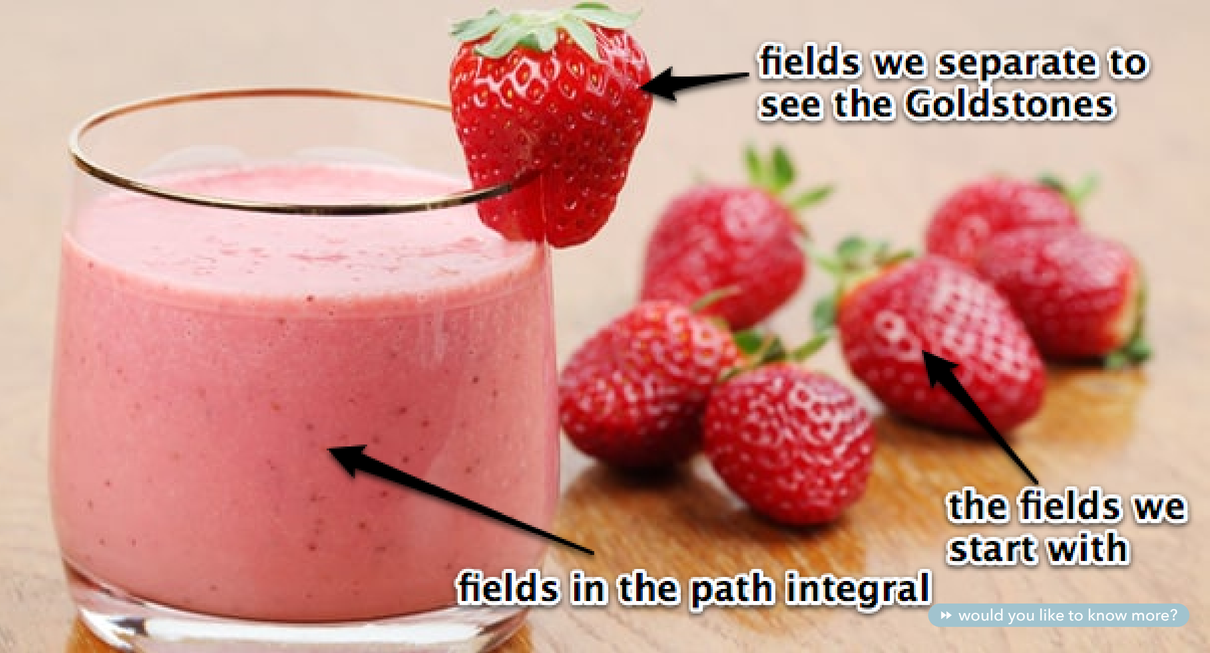
The grandpotential in a generic field is

$$\Omega(\mu) = 4\pi N I_{0,0}\left((\mu - V(r))^{5/2} + \frac{5}{64}\frac{\partial_i V(r)\,\partial_i V(r)}{\sqrt{\mu - V(r)}} - \frac{25}{48}\partial_i\partial_i V(r)\sqrt{\mu - V(r)} + \dots \right)$$

and the conformal dimension of the lowest operator follows

$$\frac{\Delta(Q)}{N} = 0.8313\dots\left(\frac{Q}{N}\right)^{4/3} + 0.2631\dots\left(\frac{Q}{N}\right)^{2/3} + \dots$$

UNIVERSAL TERM: INTEGRATE ALL BUT ONE



WAS IT WORTH IT?

FINAL RESULT

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{1/2} + \ldots$$

$$- 0.0937\ldots$$

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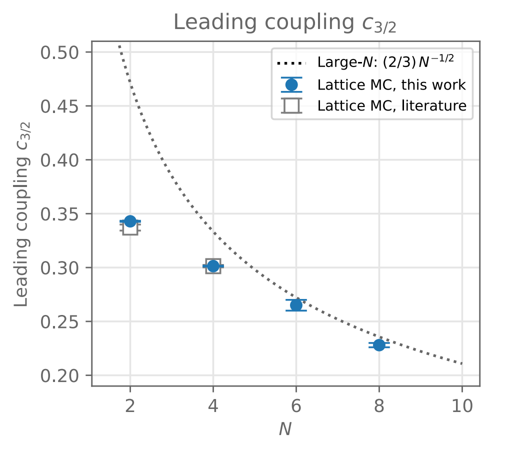

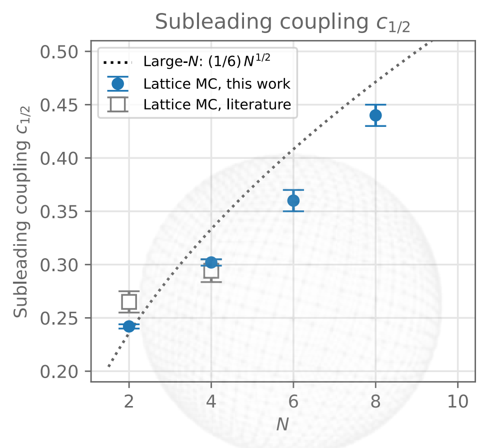

RESURGENCE AND THE LARGE CHARGE

RESULTS FROM LARGE N

O(2N) at criticality in 1 + 2 dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.

$$\begin{cases} F_{\Sigma}(Q) = \mu Q + N\zeta(-\frac{1}{2}|\Sigma, \mu), \\ \mu\zeta(\frac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}. \end{cases}$$

RESULTS FROM LARGE N

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The free energy per DOF $f(\hat{q}) = F/(2N)$ is

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)), \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

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$$\zeta(s|\Sigma, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t} \operatorname{Tr}\left(e^{\triangle t}\right).$$

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Large $\hat{q}$ is large μ and is small t. The classical Seeley-de Witt problem:

$$\mathrm{Tr}\left(e^{\triangle t}\right) \sim \frac{V}{4\pi t}\left(1 + \frac{R}{12} t + \dots\right).$$

THE TORUS

As a warm-up: $\Sigma = T^2$.

$$\mathrm{spec}(\triangle) = \left\{ -\frac{4\pi^2}{L^2}\left(k_1^2 + k_2^2\right) \mid k_1, k_2 \in \mathbb{Z} \right\}.$$

It follows that the heat kernel trace is the square of a theta function:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}(k_1^2 + k_2^2)t} = \left[\theta_3\left(0, e^{-\frac{4\pi^2 t}{L^2}}\right)\right]^2.$$

We are interested in the small-t limit: we Poisson-resum the series:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1 + \sum_{k \in \mathbb{Z}} e^{-\frac{k^2 L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{\|\mathbf{k}\|^2 L^2}{4t}}\right)$$

THE TORUS

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|T^2,\mu) = \frac{L^2\mu^3}{12\pi}\left(1 + \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\mu L}}{\|\mathbf{k}\|^2\mu^2 L^2}\left(1 + \frac{1}{\|\mathbf{k}\|\mu L}\right)\right).$$

Free energy

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}\left(1 - \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\sqrt{4\pi\hat{q}}}}{8\|\mathbf{k}\|^2\pi\hat{q}} + \dots\right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/\hat{q}$ and $e^{-\sqrt{4\pi\hat{q}}}$.
- non-perturbative effects more important than the “usual” instantons $\mathcal{O}\left(e^{-\hat{q}}\right)$

THE SPHERE

On the two sphere $\text{spec}(\triangle) = \{-\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell + 1$.

Again, we use Poisson resummation

$$\text{Tr}\left(e^{\triangle t}\right)e^{-t/4} = \sum_{\ell \geq 0} (2\ell + 1)e^{-(\ell+1/2)^2 t} \sim \frac{1}{t}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\left(1 - 2^{1-2n}\right)}{n!} B_{2n} t^n$$

The series is asymptotic: the Seeley-de Witt coefficients diverge like n!:

$$a_n = \frac{(-1)^{n+1}\left(1 - 2^{1-2n}\right)}{n!} B_{2n} \sim \frac{2n^{1/2}}{\pi^{5/2+2n}} n!.$$

this divergence is reflected in the existence of non-perturbative corrections.

BOREL RESUMMATION

BOREL TRANSFORM

We need to make sense of the divergent series and the imaginary terms.

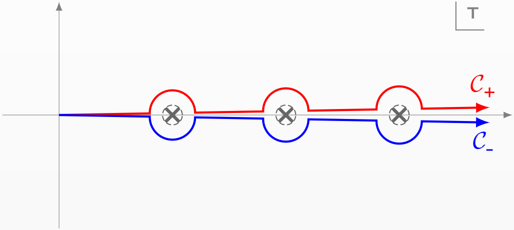
$$H(t) = \sum_{n \geq 0} a_n t^n$$

$$\hat{H}(\tau) = \sum_{n \geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

LATERAL TRANSFORM

If there are poles on the real positive axis there is an ambiguity



$$s_\pm(H)(t) = s(H)(t) = \int_{\mathcal{C}_\pm} w^b e^{-w} \hat{H}(tw^\beta) \frac{dw}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.

MORE INGREDIENTS

WORLDLINE INTERPRETATION

We need a **non-perturbative interpretation** of these exponential terms.

We read the heat kernel as the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \triangle$, i.e. a **free quantum particle moving on $\mathbb{R} \times \Sigma$**.

We can write the partition function as a **path integral**

$$\mathrm{Tr}\left(e^{(\partial_0^2 + \triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X \, e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t} \int_0^1 d\tau \, g_{\mu\nu} \dot{X}^\mu(\tau) \dot{X}^\nu(\tau)$$

A TRANSSERIES FROM GEODESICS

In the limit $t \to 0$ the path integral localizes on a sum over all the closed geodesics γ.

For each geodesic a perturbative series in t, weighted by $e^{-\ell(\gamma)^2/(4t)}$

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

$$= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma \in \text{closed geodesics}} e^{-\frac{\ell(\gamma)^2}{4t}}\, t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,$$

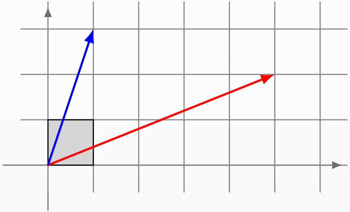
the b_γ depend on the geometry.

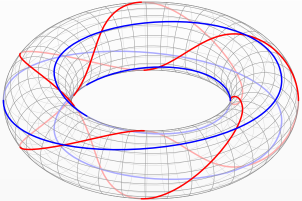
This is precisely the same structure predicted by resurgence.

Now we have a geometric interpretation.

THE TORUS

In the case of the torus, closed geodesics are labelled by two integers (k_1, k_2)





The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h \, e^{-\frac{1}{4t}\int_0^1 d\tau (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t}\partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$

THE TORUS

Now we can write the result of the path integral

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X \, e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{cl}} \int_{h(1)=h(0)=0} e^{-S[X_{cl}]-S[h]}$$

$$= \mathcal{N} L^2 \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 (k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h \, e^{-S[h]},$$

$$= \frac{L^2}{4\pi t}\left[1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 \|k\|^2}{4t}}\right]$$

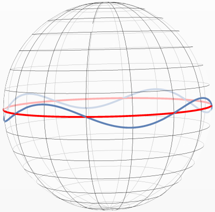
This is exactly what we had found before just by looking at the spectrum. Now we can understand the non-perturbative effects in terms of closed geodesics.

THE SPHERE

Closed geodesics on the sphere go around the equator k times

THE SPHERE

Closed geodesics on the sphere go around the equator k times



We need to sum over the fluctuations h_φ and h_θ

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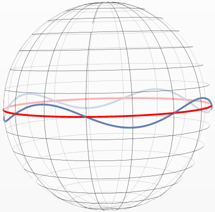

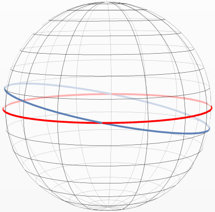

We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator

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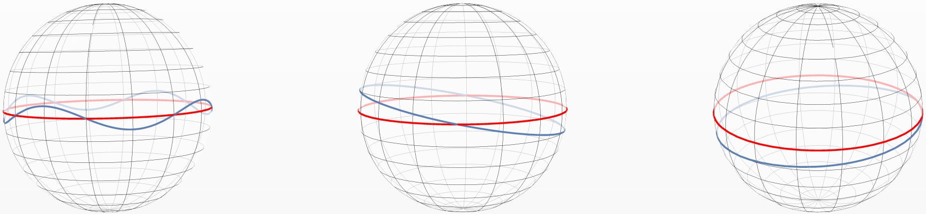

We need to sum over the fluctuations h_φ and h_θ

There is a zero mode because we can rotate the equator

And an instability because we can slide off

BACK TO RESURGENCE

Putting it **all together**, the non-trivial geodesics give

$$\pm 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\in\mathbb{Z}}|k|e^{-\frac{k^2\pi^2}{t}}$$

The one-loop result **perfectly cancels** the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr}\left(e^{(\triangle-\frac{1}{4})t}\right) = s_{\pm}(H)(t) \mp 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\geq 1}(-1)^k k e^{-\frac{k^2\pi^2}{t}} = \mathrm{Re}[s_{\pm}(H)(t)]$$

BACK TO RESURGENCE

We can write the **exact expression** for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \mathrm{Re}\left[\frac{2rm^2}{\pi}\int_0^\infty dy\,\frac{K_2(2mry)}{y\sin(y)}\right] = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

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As a numerical test, we can compare with the convergent small-charge expansion ($\hat{q} \approx 0.6$)

$$r\omega(mr = 0.4)\Big|_{\text{small charge}} = 0.012\,777\,296\,63\ldots$$

$$r\omega(mr = 0.4)\Big|_{\text{resurgence}} = 0.012\,777\,297\,69\ldots$$

OPTIMAL TRUNCATION

LESSONS FROM LARGE N

Let's go back to the EFT.

The effective action is identified with the asymptotic expansion: the **grand potential** is the value of the **action at the minimum** $\chi = \mu t$:

$$\omega(\mu) = L_{EFT}\Big|_{\chi=\mu t}$$

where

$$L_{EFT} = \omega_0 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{3/2} + \omega_1 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{1/2} + \dots,$$

In general the **coefficients are unknown**

BUT

Now we have a **geometric understanding** of the non-perturbative effects

LESSONS FROM LARGE N

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is
   a particle of mass μ going around the equator.

A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n \geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi\kappa f_0^{(0)} \sqrt{Q}} \sum_{n \geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \dots$$

(we used $\mu = 3 f_0^{(0)} \sqrt{Q}/2 + \dots$)

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LESSONS FROM LARGE N

- The **controlling parameter** for the non-perturbative effects $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ is fixed by the **leading term** in the 1/Q expansion.
- The non-perturbative coefficient $e^{-3\pi\kappa f_0^{(0)}\sqrt{Q}}$ fixes the **large-n behavior** of the perturbative series $f_n^{(0)}$.

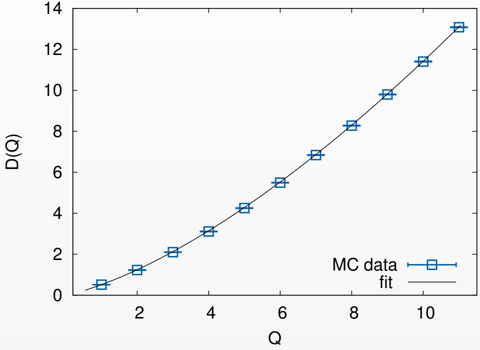
$$f_n^{(0)} \sim (2n)!(3\pi\kappa f_0^{(0)})^{-n}$$

We don't know enough for a Borel resummation, but we can estimate an optimal trucation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi\kappa f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$

CAN WE UNDERSTAND THE LATTICE RESULTS NOW?



In $O(2)$, $f_0^{(0)} \approx 0.301(3)$
so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\pi\sqrt{Q}}\right)$.

Lattice: Best fit with $N = 3$ terms.
At $Q = 1$ the error is $\approx 6 \times 10^{-2}$; at $Q = 11$ the error is $\approx 5 \times 10^{-5}$.

Resurgence: $\sqrt{10} \approx 3.16$
$e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$.

WHAT HAS HAPPENED?

- The large-charge expansion of the Wilson–Fisher point is **asymptotic**
- In the **double-scaling** limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion
- We can **Borel**-resum the expansion
- We have a **geometric interpretation for the non-perturbative effects**
- We can use this geometric interpretation also in the **finite-N** case
- We obtain an **optimal truncation** and estimate of the error
- The results are **consistent with lattice simulations**

CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Precise and **testable predictions**.
- Qual(nt)itative control of the **non-pertubative** effects.
- **CFT constraints**: perturbative/non-perturbative **interplay**.
- Remarkable agreement with **lattice**.

ORDER N°

The order N^0 terms are

$$S^\theta[\hat{\sigma}, \hat{\lambda}] = \int dt\, d\Sigma \left((D_\mu \hat{\sigma})^* (D^\mu \hat{\sigma}) + (\mu^2 + \hat{\lambda})\hat{\sigma}^* \hat{\sigma} + \frac{\hat{\lambda} v(\hat{\sigma} + \hat{\sigma}^*)}{(N-1)^{1/2}} \right) + \frac{1}{2} \int dx_1\, dx_2\, \hat{\lambda}(x_1)\hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.

At low energies we can approximate the non-local term as

$$\int dt\, d\Sigma\, \hat{\lambda}(x)^2 \zeta(2|\theta, \Sigma, \mu) \approx \frac{V}{2\mu} \int dt\, d\Sigma\, \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

ORDER N°

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4\mu^2) & \mu\omega \\ -\mu\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \ldots = 0 \qquad\qquad \omega^2 + 8\mu^2 + \frac{3}{2}p^2 + \ldots = 0$$

This is the conformal Goldstone that we have seen in the EFT.

Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\ldots$$

This is **universal**. Does not depend on N or Q.

HIGHER ORDERS

There are infinite non-local terms

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx_1 \ldots dx_n \, \hat{\lambda}(x_1)\ldots\hat{\lambda}(x_n) P(x_1, \ldots, x_n)$$

At low energy they are approximated by

$$S_{nl} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int dx \, \hat{\lambda}(x)^n C_n$$

HIGHER ORDERS

There is only one scale, the charge density ρ = Q/V. We must have

$$C_n = \rho^{3/2-n} c_n$$

So

$$S_{nl} = Q^{3/2} \sum_{n=3}^{\infty} \frac{c_n}{n(N-1)^{n/2-1}} \int dx\, \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by 1/N.

back to the talk