# N=2 SUSY at large R-charge

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# What happened?

We started from a conformal field theory (CFT).
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.

In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

would you like to know more?

# And Now for Something Completely Different

The $O(2)$ model has a isolated vacuum.
What happens when there is a flat direction?

Many known examples of (non-Lagrangian) $\mathcal{N} \geq 2$ SCFT in four dimensions.

Coulomb branch with a dimension-one moduli space: all the physics is encoded in a single operator $\mathcal{O}$ and every chiral operator is just $\mathcal{O}^n$.

We will write an effective action for a canonically-normalized dimension-one vector multiplet $\Phi$.

# Effective action

We have a single vector multiplet. The kinetic term is just

$$L_k = \int d^4\theta \, \Phi^2 + \text{c.c.} = |\partial\varphi|^2 + \text{fermions} + \text{gauge fields}$$

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The coefficient $a$ fixes the a-anomaly of the EFT. It has to match the anomaly in the UV.

**Claim**: at large R-charge this action is all you need for any $\mathcal{N} = 2$ theory (with one-dimensional moduli space).

# Observables

Three-point function of the Coulomb branch operators

$$\left\langle \mathcal{O}^{n_1}(x_1)\mathcal{O}^{n_2}(x_2)\bar{\mathcal{O}}^{n_1+n_2}(x_3) \right\rangle = \frac{C^{n_1,n_2,n_1+n_2}}{|x_1 - x_3|^{2n_1\Delta}|x_2 - x_3|^{2n_2\Delta}}$$

The OPE of $\mathcal{O}$ with itself is regular, so we can set $x_2 = x_1$ and the three-point function is actually a two-point function.

$$C^{n,n-n,n} = |x_1 - x_2|^{2n\Delta} \left\langle \mathcal{O}^{n}(x_1)\bar{\mathcal{O}}^{n}(x_2) \right\rangle = e^{q_n - q_0} = G_{2n}$$

$Q = n\Delta$ is the controlling parameter (it's the R-charge)

# Two-point function

$$\left\langle \Phi^n(x_1)\bar{\Phi}^n(x_2)\right\rangle = \int D\varphi\, \varphi^n(x_1)\bar{\varphi}^n(x_2) e^{-S_k}$$

We can just pull the sources in the action and minimize

$$S_k + S_{\text{sources}} \propto k_0 + \int d^4x \left[ \partial_\mu\varphi\, \partial_\mu\bar{\varphi} - Q\log\varphi\, \delta(x - x_1) - Q\log\bar{\varphi}\,\delta(x - x_2)\right]$$

At the minimum:

$$q_n = k_1 Q + k_0 + Q\log(Q) + \left(a + \frac{1}{2}\right)\log(Q) + \mathcal{O}\left(Q^0\right)$$

Corrections from **quantum fluctuations** in the path integral as a series in $1/Q$.
**No other tree-level terms**.

# Two-point function: quantum corrections

$1/Q$ **is the loop-counting parameter** because we are expanding around a VEV that depends on $Q$.
Sum of a ground state piece and a series in $1/Q$.

$$q_n = k_0 + k_1 Q + Q \log(Q) + \left(\alpha + \frac{1}{2}\right) \log(Q) + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

The interaction comes from the WZ term and can **only depend on** $\alpha$.

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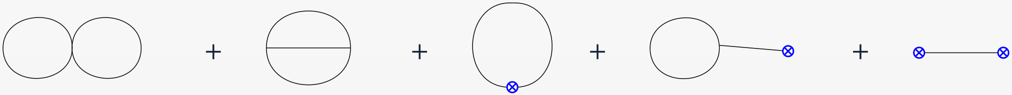
Compute order-by-order



$$k_1(\alpha) = \frac{1}{2}\left(\alpha^2 + \alpha + \frac{1}{6}\right)$$

# Integrability to the rescue

There is a better way.
Using **localization** one finds that the $q_n$ satisfy arXiv:0910.4963

$$\partial_\tau \partial_{\bar{\tau}} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}$$

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$$q_n(\tau,\bar{\tau}) = B(\tau,\bar{\tau}) + nA(\tau,\bar{\tau}) + n\Delta\log(n\Delta) + \left(\alpha + \frac{1}{2}\right)\log(n\Delta) + \sum_{m=1}^{\infty}\frac{k_m(\alpha)}{n^m}$$

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# Toda lattice + large charge

The Toda-lattice equation turns into a pair of coupled Liouville-like equations for A and B and a **difference equation** for the $\tau$-independent part.

We can actually solve the recursion relation, using the value of $k_1(\alpha)$ found at one loop.

$$q_n = B(\tau, \bar{\tau}) + nA(\tau, \bar{\tau}) + \log\big(\Gamma(n\Delta + \alpha + 1)\big)$$

The log term is **universal**, only depends on $\alpha$.

We have **completely resummed** the $1/Q$ expansion.

# Recursion relation

In terms of the generator $\mathcal{O}$ of the Coulomb branch we have:

$$\left\langle \mathcal{O}^n(x_1)\bar{\mathcal{O}}^n(x_2)\right\rangle = C_n(\tau,\bar{\tau})\frac{\Gamma(n\Delta + \alpha + 1)}{|x_1 - x_2|^{2n\Delta}}$$

The coefficient $C_n$ depends on the normalization of $\mathcal{O}(x)$.

Crucial: **This form is valid for any $\mathcal{N} = 2$ SCFT with dimension-one Coulomb branch. Including non-Lagrangian theories.**

# The linear term

In fact we can do better in the case of SQCD.

$$C_n = e^{nA+B} + \mathcal{O}\left(e^{-\kappa\sqrt{n}}\right),$$

and the Toda-lattice equation reduces to the Liouville equation for A:

$$\partial\bar{\partial}A = 8e^{A}.$$

The general solution depends on two arbitrary functions

$$e^{A} = \frac{\partial f \bar{\partial}\tilde{f}}{(1 - 4f\tilde{f})^{2}}.$$

We need one more ingredient: **S-duality**

# Back to Liouville

We look for the solution to the Liouville equation

$$\partial_\sigma \partial_{\bar\sigma} \hat{A}(\sigma, \bar\sigma) = 8 e^{\hat{A}(\sigma,\bar\sigma)},$$

where $e^{\hat{A}(\sigma,\bar\sigma)}$ is a modular form of weight $(2, 2)$.

Transforming back to the $\tau$ coordinate we find

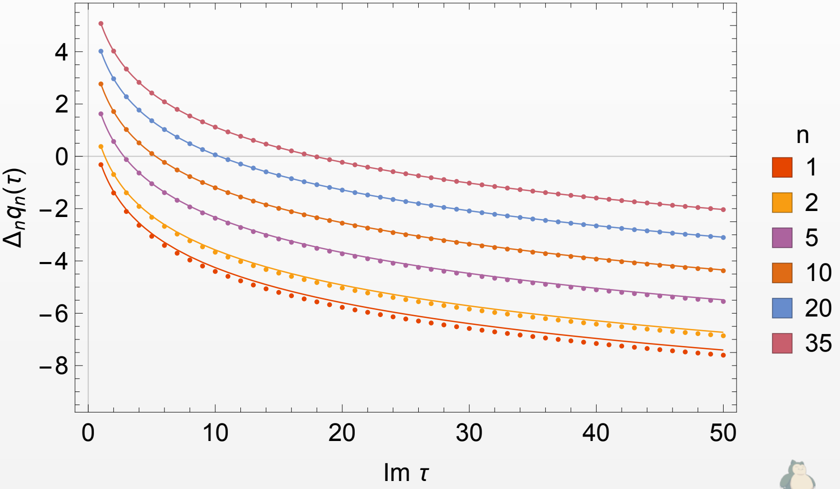
$$A(\tau, \bar\tau) = \log\left(\frac{1}{4\left(2\,\mathrm{Im}(\tau) + 4/\pi \log(2)\right)^2}\right) + \mathcal{O}\left(e^{2\pi i \tau}\right).$$

# Comparison with localization

How well does this work?

For the special case of SU(2) SQCD with $N_f = 4$ we can compare with localization.

arXiv:1602.05971

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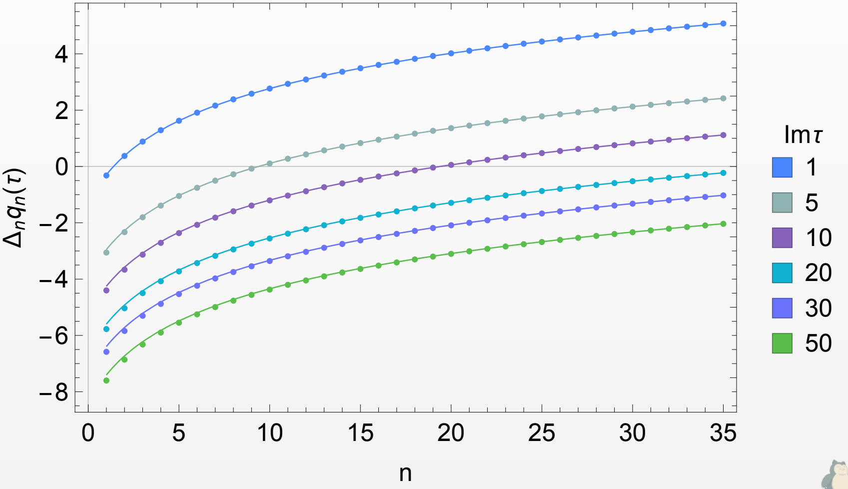
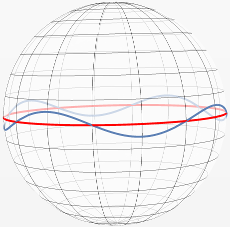



# Adding instantons

We can do better.

We have resummed the 1/Q expansion around one vacuum.
Exponential corrections coming from the **next saddle in the path integral**.
BPS particles going around the equator of the three-sphere.



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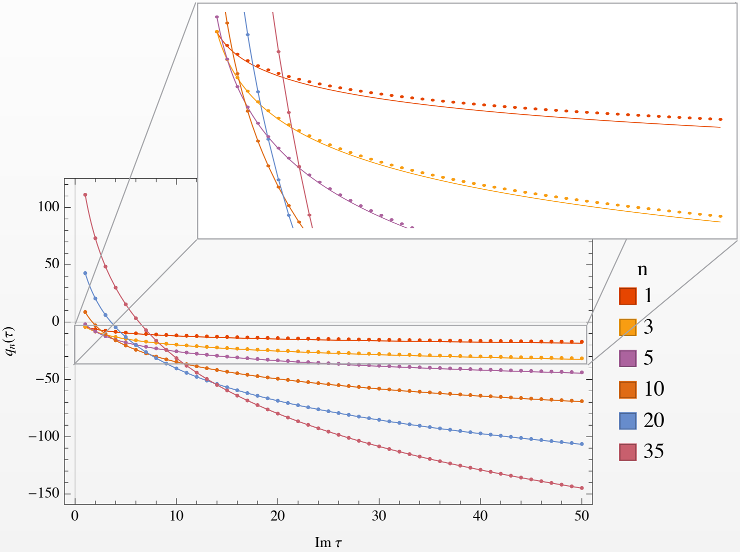


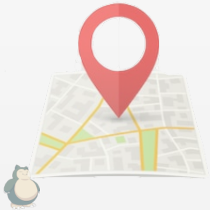

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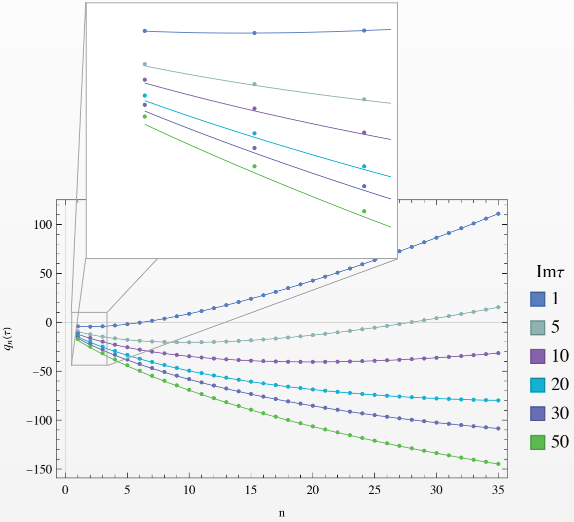
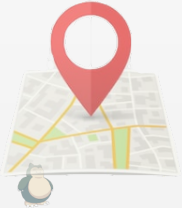
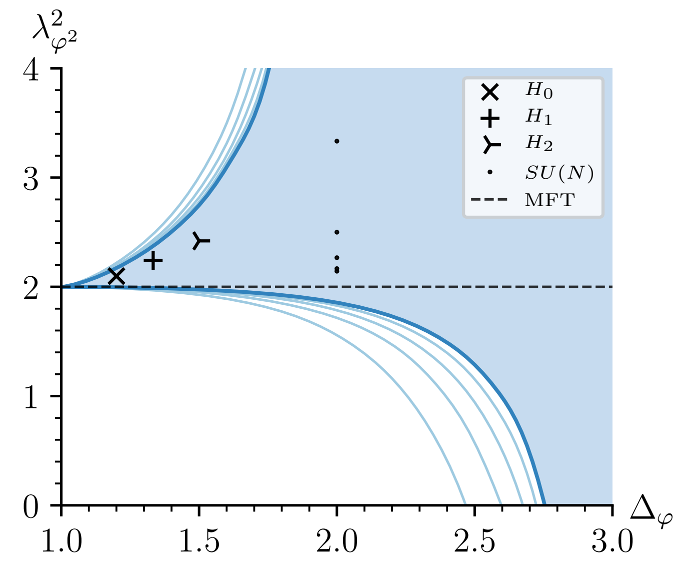


# Comparison with bootstrap

For strongly coupled theories one can use bootstrap to place bounds on the three-point coefficients with $n = 1$.
This is the worst possible situation for us. And still...

# Comparison with boostrap




Taken from arXiv:2006.01847

# Conclusions

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Qual(nt)itative control of the **non-pertubative** effects.
- Compute the CFT data.
- Very good agreement with **lattice** (supersymmetry, large N).
- Precise and **testable predictions**.