The O(N) vector model at large charge: EFT, large N and resurgence.

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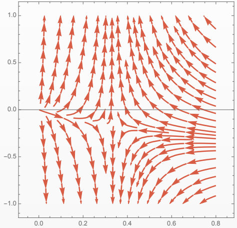
arXiv:1505.01537, arXiv:1610.04495, arXiv:1707.00711, arXiv:1804.01535, arXiv:1902.09542, arXiv:1905.00026,arXiv:1909.02571, arXiv:1909.08642, arXiv:2003.08396, arXiv:2005.03021, arXiv:2008.03308, arXiv:2010.07942, arXiv:2102.12488, arXiv:2103.05642 and more to come…

Who’s who

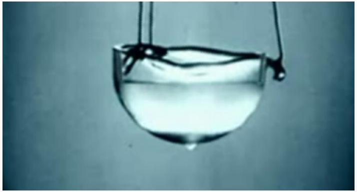
L. Álvarez Gaumé (SCGP and CERN);
D. Banerjee (Calcutta);
S. Chandrasekharan (Duke);
S. Hellerman (IPMU);
S. Reffert, N. Dondi, I. Kalogerakis , V. Pellizzani (AEC Bern);
F. Sannino (CP3-Origins and Napoli);
M. Watanabe (Weizmann).

Why are we here? Conformal field theories

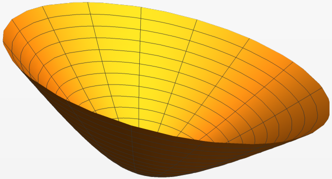
extrema of the RG flow



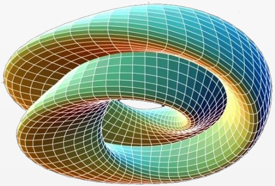
critical phenomena



quantum gravity



string theory



Why are we here? Conformal field theories are hard

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

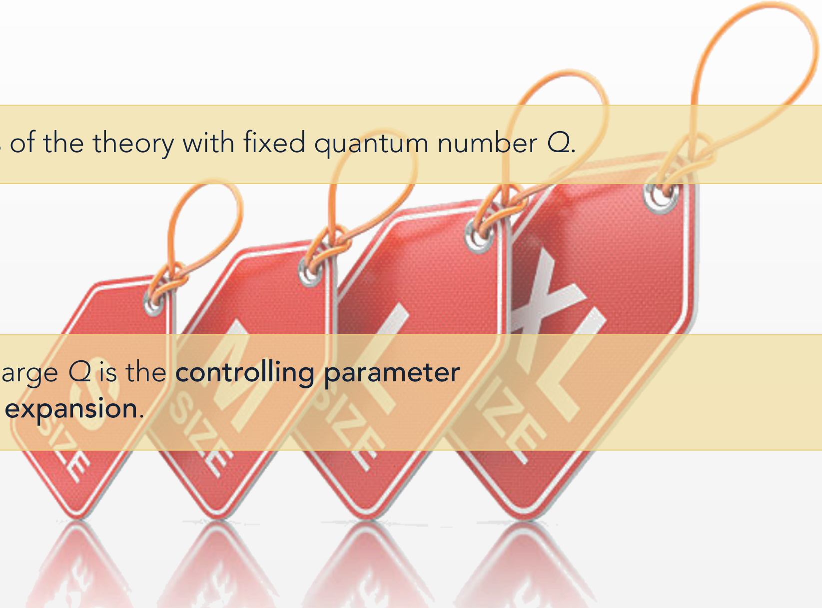
Why are we here? Conformal field theories are hard

In presence of a **symmetry** there can be **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

The idea

Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.



no bootstrap here!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
(However, arXiv:1710.11161).

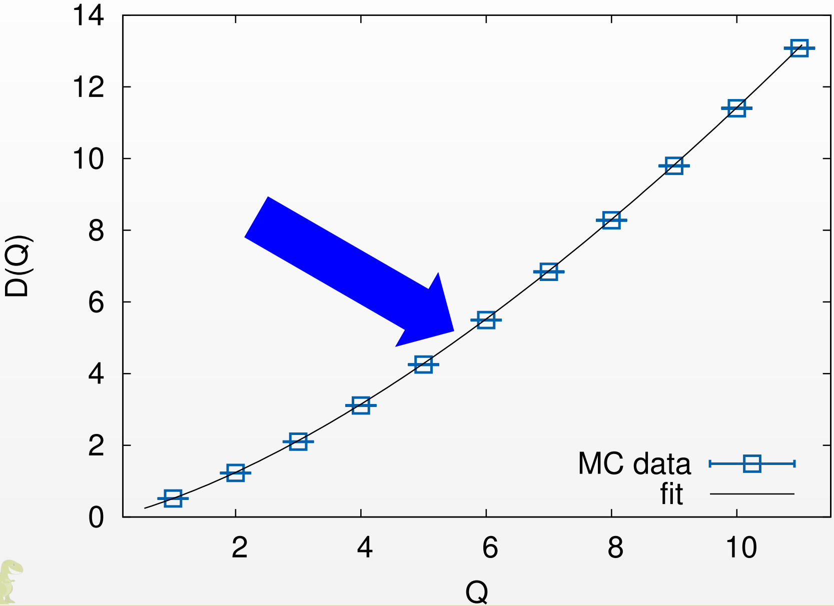
Concrete results

We consider the $O(N)$ vector model in three dimensions. In the IR it flows to a conformal fixed point [Wilson & Fisher].

We find an explicit formula for the dimension of the lowest primary at fixed charge:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

Summary of the results: O(2)



Scales

We want to write a Wilsonian effective action.

Choose a cutoff Λ, separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \, e^{iS(\phi_H, \phi_L)}$$

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too hard

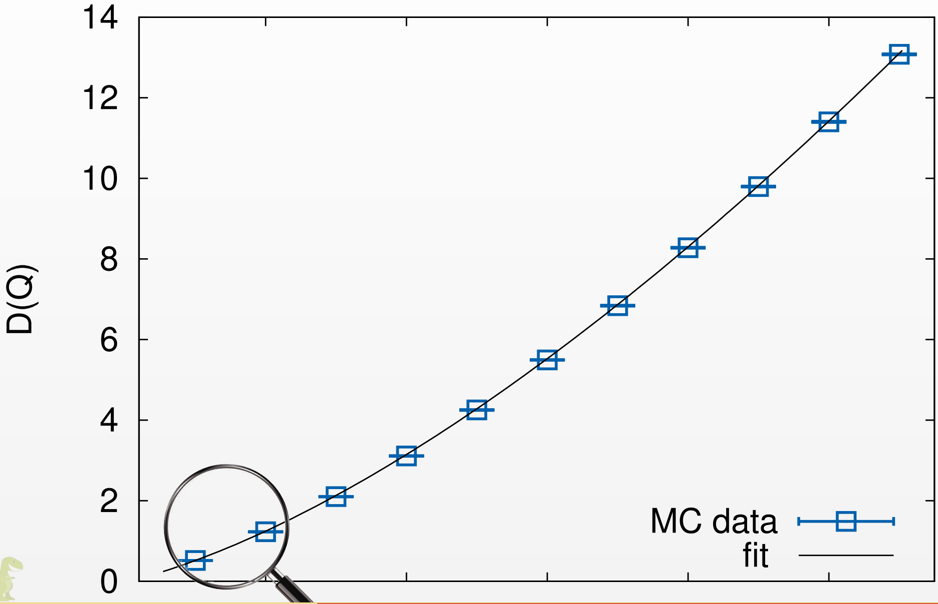
Scales

- We look at a finite box of typical length R
- The $U(1)$ charge Q fixes a second scale $\rho^{1/2} \sim Q^{1/2}/R$

$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

Too good to be true?

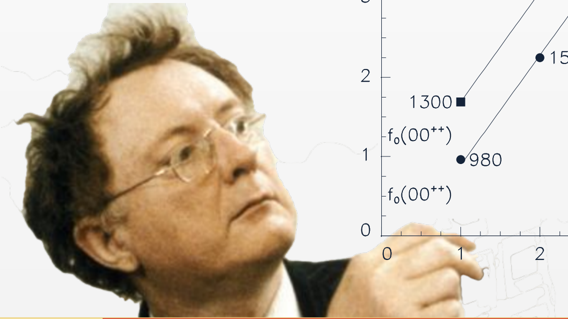


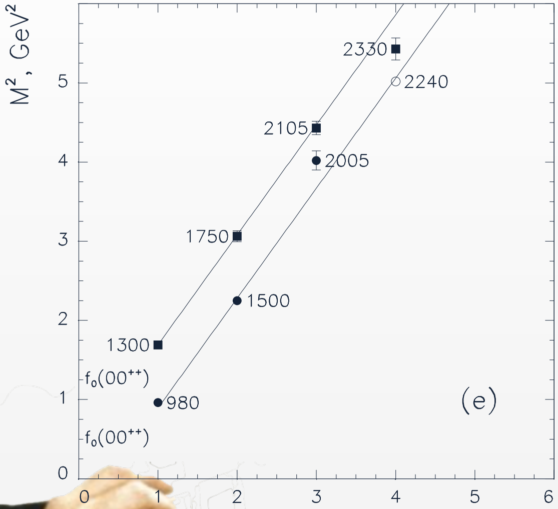
Too good to be true?

Think of Regge trajectories.
The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but *experimentally* everything works so well at small J that String Theory was invented.





Too good to be true?

The unreasonable effectiveness

of the **large charge expansion**.

Selected topics in the LQNE

- O(2) model [S.Hellerman, DO, S.Reffert, M.Watanabe] [A.Monin, D.Pirtskhalava, R.Rattazzi, F.K. Seibold]
- O(N) model [L.Álvarez-Gaumé, O.Loukas, DO, S.Reffert]
- gravitational systems [O.Loukas, DO, S.Reffert, D. Sarkar], [de la Fuente], [S.-F. Guo, H-S Liu, H.Lu, Y.Pang]
- large N [L.Álvarez-Gaumé, DO, S.Reffert], [S.Giombi, J.Hyman]
- ε double-scaling [G.Badel, G.Cuomo, A.Monin, R.Rattazzi], [G.Arias-Tamargo, D.Rodriguez-Gomez, J.G. Russo] [O.Antipin, J.Bersini, F.Sannino, Z.-W.Wang, C.Zhang] [I.Jack, T.Jones]
- non-relativistic CFTs [S.Kravec, S.Pal], [S.Hellerman, I.Swanson], [S.Favrod, DO, S.Reffert], [DO, S.Reffert, V.Pellizzani]
- $\mathcal{N} = 2$ [S.Hellerman, S.Maeda], [S.Hellerman, S.Maeda, DO, S.Reffert, M.Watanabe], [A.Bourget, D.Rodriguez-Gomez, J.G.Russo], [A.Grassi, Z.Komargodski, L.Tizzano]
- bootstrap [D.Jafferis, A.Zhiboedov]

Today's talk

The EFT for the $O(2)$ model in $2+1$ dimensions

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The EFT for the $O(2)$ model in $2+1$ dimensions

- An effective field theory (EFT) for a CFT.
- The physics at the saddle.
- State/operator correspondence for anomalous dimensions.

Today’s talk

The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

Today's talk

The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

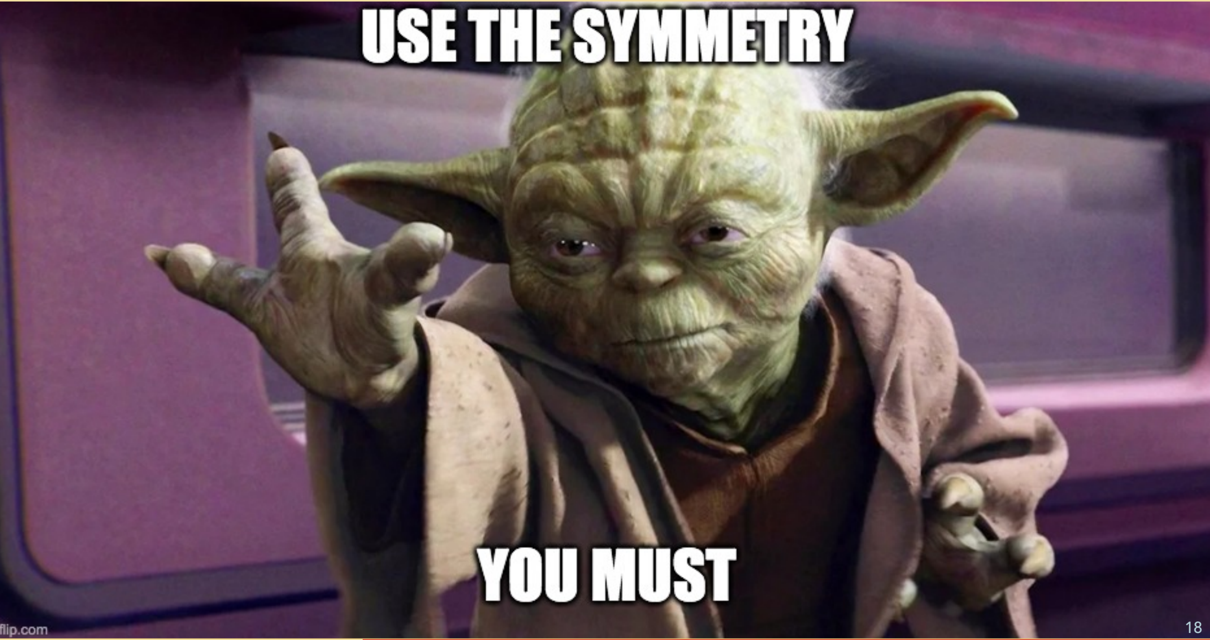
Use resurgence to reach small charge

- Borel resum the double-scaling $Q \to \infty$, $N \to \infty$ limit
- geometric interpretation of non-perturbative effects
- general structure of the corrections in the EFT

PARENTAL ADVISORY

EXPLICIT CONTENT

An EFT for a CFT



The $O(2)$ model

The simplest example is the Wilson–Fisher (WF) point of the $O(2)$ model in three dimensions.

- Non-trivial fixed point of the ϕ^4 action

$$L_{UV} = \partial_\mu \phi^* \partial_\mu \phi - u(\phi^* \phi)^2$$

- Strongly coupled
- In nature: ^4He.
- Simplest example of spontaneous symmetry breaking.
- Not accessible in perturbation theory. Not accessible in $4 - \varepsilon$. Not accessible in large N.
- Lattice. Bootstrap.

Charge fixing

We assume that the $O(2)$ symmetry is not accidental.

We consider a subsector of fixed charge Q.
Generically, the classical solution at fixed charge breaks spontaneously $U(1) \to \emptyset$.

We have one Goldstone boson χ.

An action for χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \, \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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We want to describe a CFT: we can dress with a dilaton

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \, \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \, \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken $U(1)$, the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

Linear sigma model

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

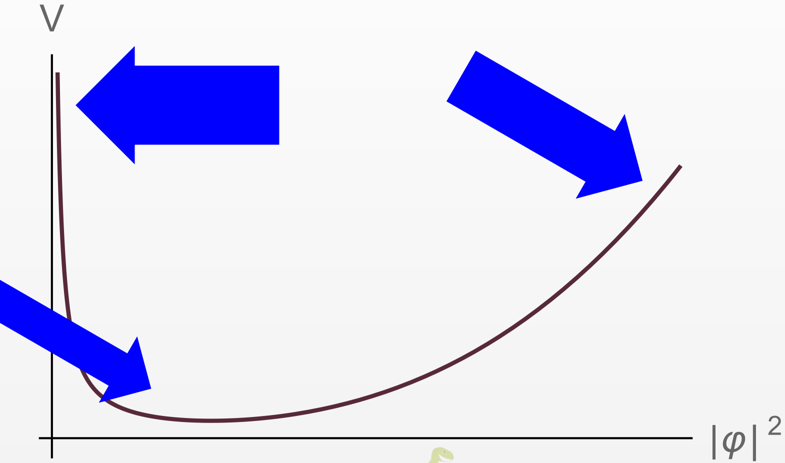
We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.
Scale invariance is manifest.
The field φ is some complicated function of the original ϕ.

Centrifugal barrier

The $O(2)$ symmetry acts as a shift on χ.
Fixing the charge is the same as adding a centrifugal term $\propto \frac{1}{|\varphi|^2}$.



Ground state

We can find a fixed-charge solution of the type

$$\chi(t,x) = \mu t \qquad \sigma(t,x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \dots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2} V Q^{3/2} + c_{1/2} R V Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

Fluctuations

The fluctuations over this ground state are described by two modes.

- A universal "conformal Goldstone". It comes from the breaking of the $U(1)$.

$$\omega = \frac{1}{\sqrt{2}} p$$

- The massive dilaton. It controls the magnitude of the quantum fluctuations. All quantum effects are controled by $1/Q$.

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

Non-linear sigma model

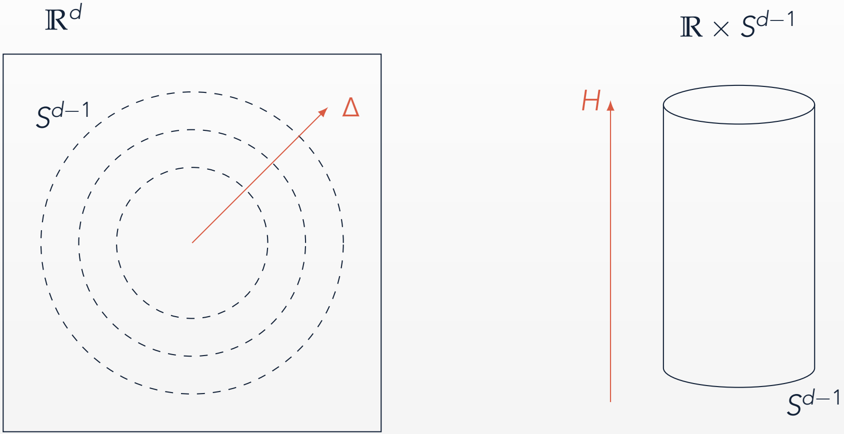
Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi\, \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi\, \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.
All other terms are suppressed by powers of $1/Q$.

State-operator correspondence

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

Conformal dimensions

We know the energy of the ground state.
The leading quantum effect is the Casimir energy of the conformal Goldstone.

$$E_G = \frac{1}{2\sqrt{2}} \zeta\left(-\tfrac{1}{2} | S^2\right) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the conformal dimension of the lowest operator of charge Q in the $O(2)$ model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

What happened?

We started from a CFT.
There is no mass gap, there are no particles, there is no Lagrangian.

We picked a sector.

In this sector the physics is described by a semiclassical configuration plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a simple EFT.
We are in a strongly coupled regime but we can compute physical observables using perturbation theory.

▸▸ would you like to know more?

Large N vs. Large Charge

The model

ϕ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int \mathrm{d}t\,\mathrm{d}\Sigma \left[g^{\mu\nu} (\partial_\mu \varphi_i)^* (\partial_\nu \varphi_i) + r \varphi_i^* \varphi_i + \frac{u}{2} (\varphi_i^* \varphi_i)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned to $R/8$.
We compute the partition function at fixed charge

$$Z(Q_1, \dots, Q_N) = \operatorname{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int \mathrm{d}\Sigma\, j_i^0 = i \int \mathrm{d}\Sigma \left[\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i \right].$$

Dimensions of operators of fixed charge Q on $\mathbb{R}^3$ (state/operator):

$$\Delta(Q) = -\frac{1}{\beta} \log Z_{S^2}(Q).$$

Fix the charge

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i} \operatorname{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$\begin{aligned} Z_\Sigma(Q) &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i\, e^{-S[\varphi]} \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i\, e^{-S^\theta[\varphi]} \end{aligned}$$

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Effective actions

The covariant derivative approach:

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u(\varphi_i^* \varphi_i)^2 \right)$$

where

$$\begin{cases} D_0 \varphi = \partial_0 \varphi + i\frac{\theta}{\beta} \varphi \\ D_i \varphi = \partial_i \varphi \end{cases}$$

Stratonovich transformation: introduce Lagrange multiplier λ and rewrite the action as

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int \mathrm{d}t\,\mathrm{d}\Sigma \left[\left(D_\mu^i \varphi_i \right)^* \left(D_\mu^i \varphi_i \right) + \left(\frac{R}{8} + \lambda \right) \varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = \left(\mu^2 - \frac{R}{8} \right) + \hat{\lambda}$$

Saddle point equations

The integral over the φ is Gaussian.
We can perform it, *e.g.* in terms of zeta functions.

$$\zeta(s|\Sigma,\mu) = \operatorname{Tr}\left((\nabla_\Sigma^2 - \mu^2)^{-s}\right)$$

With some massaging, we find the final equations

$$\begin{cases} F_\Sigma(Q) = \mu Q + N\zeta(-\tfrac{1}{2}|\Sigma,\mu), \\ \mu\,\zeta(\tfrac{1}{2}|\Sigma,\mu) = -\tfrac{Q}{N}. \end{cases}$$

The control parameter is actually Q/N.

Large Q/N

If $Q/N \gg 1$ we can use Weyl's asymptotic expansion.

$$\mathrm{Tr}\left(e^{\Delta_\Sigma t}\right) = \sum_{n=0}^{\infty} K_n t^{n/2-1}.$$

The zeta function is written in terms of the geometry of Σ (heat kernel coefficients)

$$\mu_\Sigma = \sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{1/2} + \frac{R}{24}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{-1/2} + \dots$$

$$\frac{F_\Sigma}{2N} = \frac{2}{3}\sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{3/2} + \frac{R}{12}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{1/2} + \dots$$

Order N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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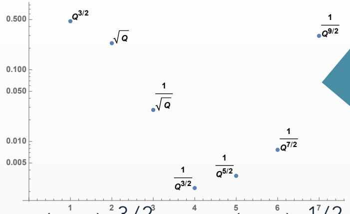

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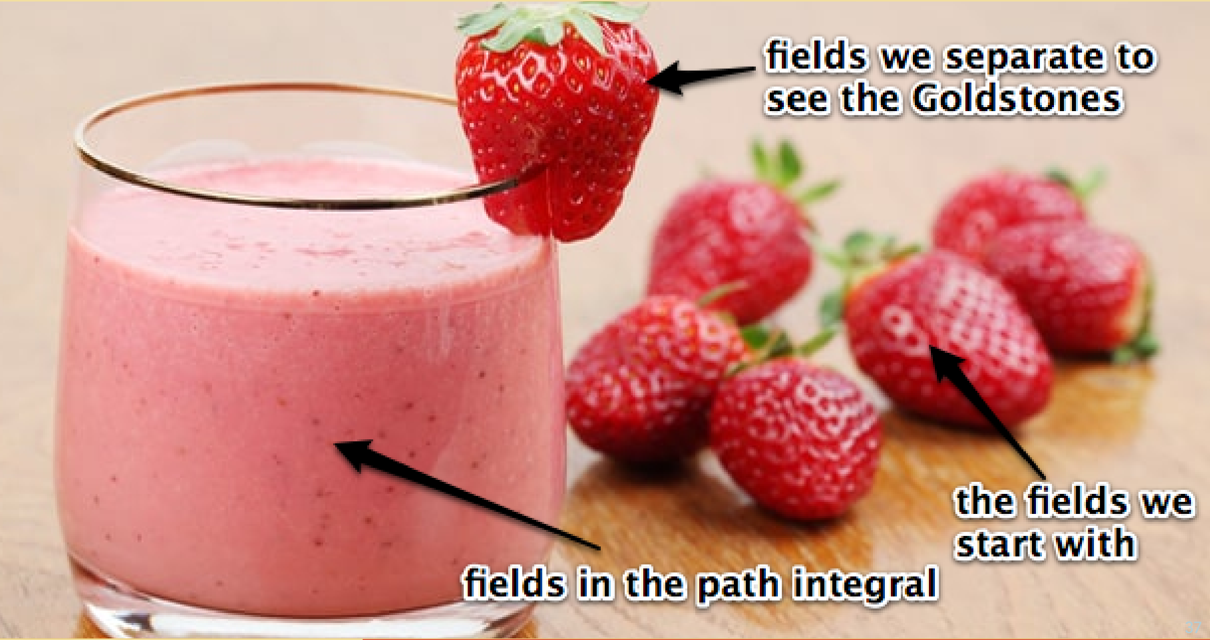
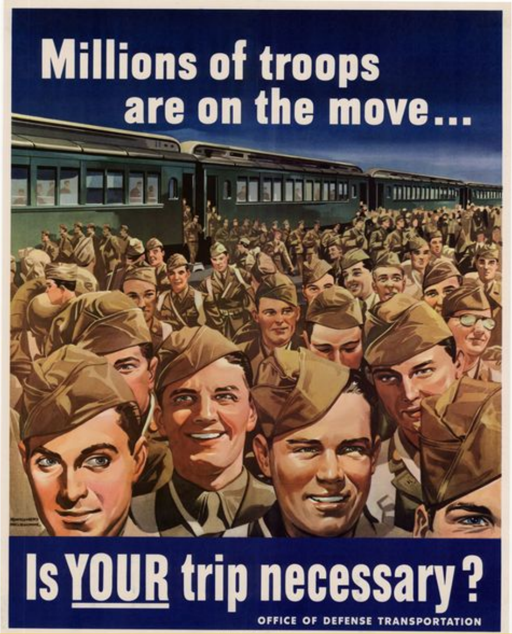
Where is the universal Goldstone?

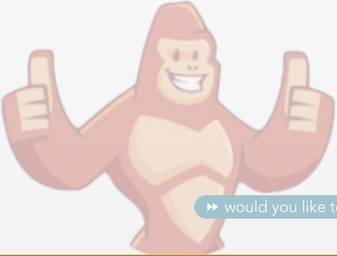


Was it worth it?



Final result

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{1/2} + \dots$$
$$- 0.0937\dots$$



would you like to know more?

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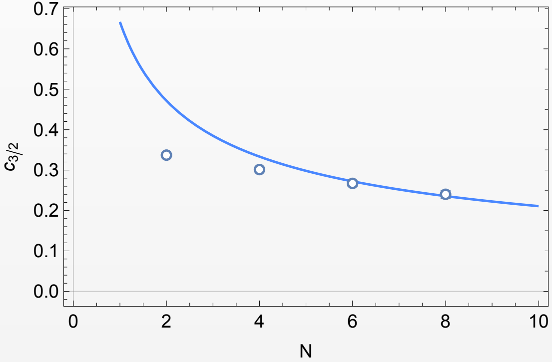

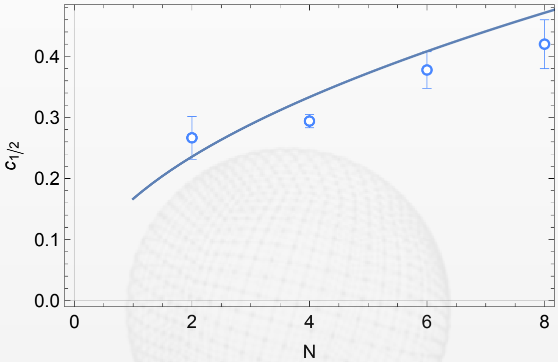

would you like to know more?

Resurgence and the large charge

Results from large N

O(2N) at criticality in $1+2$ dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.

$$\begin{cases} F_{\Sigma}(Q) = \mu Q + N \zeta(-\tfrac{1}{2}|\Sigma, \mu), \\ \mu\, \zeta(\tfrac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}. \end{cases}$$

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O(2N) at criticality in $1+2$ dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.
The free energy per DOF $f(\hat{q}) = F/(2N)$ is

$$f(\hat{q}) = \sup_{\mu}(\mu \hat{q} - \omega(\mu)), \qquad \hat{q} = \frac{d\omega(\mu)}{d\mu}, \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

Results from large N

O(2N) at criticality in $1+2$ dimensions on $\mathbb{R} \times \Sigma$. Double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.
The free energy per DOF $f(\hat{q}) = F/(2N)$ is

$$f(\hat{q}) = \sup_{\mu}(\mu \hat{q} - \omega(\mu)), \qquad \hat{q} = \frac{d\omega(\mu)}{d\mu}, \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

$\zeta(s|\Sigma, \mu)$ is the zeta function for the operator $-\triangle + \mu^2$. In Mellin representation

$$\zeta(s|\Sigma, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-\mu^2 t} \operatorname{Tr}\left(e^{\triangle t}\right).$$

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Large $\hat{q}$ is large μ and is small t. The classical Seeley–de Witt problem:

$$\operatorname{Tr}\left(e^{\triangle t}\right) \sim \frac{V}{4\pi t}\left(1 + \frac{R}{12} t + \dots\right).$$

The torus

As a warm-up: $\Sigma = T^2$.

$$\operatorname{spec}(\triangle) = \{-\frac{4\pi^2}{L^2}\left(k_1^2 + k_2^2\right) | k_1, k_2 \in \mathbb{Z}\}.$$

It follows that the heat kernel trace is the square of a theta function:

$$\operatorname{Tr}\left(e^{\triangle t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}(k_1^2+k_2^2)t} = \left[\theta_3(0, e^{-\frac{4\pi^2 t}{L^2}})\right]^2.$$

We are interested in the small-t limit.
For this reason we Poisson-resum the series:

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} h(\rho) e^{2\pi i k\rho} \, d\rho$$

$$\operatorname{Tr}\left(e^{\triangle t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1 + \sum_{k\in\mathbb{Z}}{}' e^{-\frac{k^2 L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1 + \sum_{\mathbf{k}\in\mathbb{Z}^2}{}' e^{-\frac{\|\mathbf{k}\|^2 L^2}{4t}}\right)$$

The torus

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|T^2,\mu) = \frac{L^2\mu^3}{12\pi}\left(1 + \sum_{\mathbf{k}}{}' \frac{e^{-\|\mathbf{k}\|\mu L}}{\|\mathbf{k}\|^2\mu^2 L^2}\left(1 + \frac{1}{\|\mathbf{k}\|\mu L}\right)\right).$$

Free energy

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}\left(1 - \sum_{\mathbf{k}}{}' \frac{e^{-\|\mathbf{k}\|\sqrt{4\pi\hat{q}}}}{8\|\mathbf{k}\|^2\pi\hat{q}} + \dots\right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/\hat{q}$ and $e^{-\sqrt{4\pi\hat{q}}}$.
- non-perturbative effects more important than the "usual" instantons $\mathcal{O}(e^{-\hat{q}})$

The sphere

On the two sphere $\mathrm{spec}(\triangle) = \{-\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell+1$.

Again, we use Poisson resummation

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} h(\rho) e^{2\pi i k \rho} \, d\rho$$

to rewrite the heat kernel in terms of the imaginary error function

$$\mathrm{Tr}\left(e^{\triangle t}\right) e^{-t/4} = \sum_{\ell\geq 0} (2\ell+1) e^{-(\ell+1/2)^2 t} = \frac{r^2}{t} + 2\sum_{k\in\mathbb{Z}}{}' (-1)^k \left[\frac{r^2}{t} - \frac{2k\pi r^3}{t^{3/2}} F\left(\frac{\pi r k}{t^{1/2}}\right)\right]$$

where

$$F(z) = e^{-z^2} \int_0^z dt \, e^{-t^2} = \frac{\sqrt{\pi}}{2} e^{-z^2} \operatorname{erfi}(z)$$

Sphere: asymptotic expansion

For small t

$$F(z) \sim \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^{n+1}} \left(\frac{1}{z}\right)^{2n+1}$$

and

$$\mathrm{Tr}\left(e^{(\triangle - \frac{1}{4})t}\right) \sim \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(1-2^{1-2n})}{n!} B_{2n} t^n$$

The series is asymptotic: the Seeley–de Witt coefficients diverge like $n!$:

$$a_n = \frac{(-1)^{n+1}(1-2^{1-2n})}{n!} B_{2n} \sim \frac{2n^{1/2}}{\pi^{5/2+2n}} n!.$$

this divergence is reflected in the existence of non-perturbative corrections.

Resurgence

The key idea is that we should think in terms of transseries

$$H(t) = t^{-b_0} \sum_{n\geq 0} a_n^{(0)} t^n + \sum_{k\geq 1} C_k e^{-A_k/t} t^{-b_k} \sum_{n\geq 0} a_n^{(k)} t^n,$$

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The coefficients of the non-perturbative part are encoded in the large-n behavior of the perturbative piece:

$$a_n^{(0)} \sim \sum_{k\geq 1} \frac{C_k}{2\pi i} \frac{1}{A_k^{n/\beta + b_k}} \left(a_0^{(k)} \Gamma\left(\beta n + b_k\right) + a_1^{(k)} A_k \Gamma\left(\beta n + b_k - 1\right) + \dots \right)$$

Resurgence

In our case, the a_n are

$$a_n = 4\sqrt{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma\left(n+\frac{1}{2}\right)}{(\pi k)^{2n}}.$$

Comparing the two expressions we find that for the trace of the heat kernel:

$$\beta = 1, \qquad b_k = \frac{1}{2}, \qquad A_k = (\pi k)^2, \qquad \frac{C_k}{2\pi i}a_0^{(k)} = 4(-)^k k\pi^{3/2}, \qquad a_{>0}^{(k)} = 0.$$

The series around each exponential are truncated to only one term and the non-perturbative correction to the heat kernel is

$$4i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k=1}^{\infty}(-)^k k\, e^{-(\pi k)^2/t}.$$

Borel resummation

Borel transform

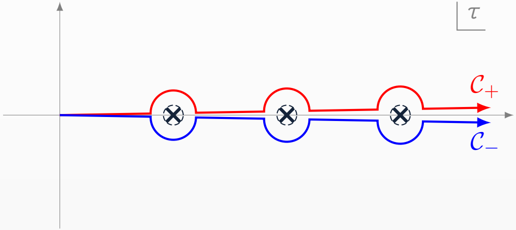
We need to make sense of the divergent series and the imaginary terms.

$$H(t) = \sum_{n\geq 0} a_n t^n \longrightarrow \hat{H}(\tau) = \sum_{n\geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

Lateral transform

If there are poles on the real positive axis there is an ambiguity



$$s_\pm(H)(t) = s(H)(t) = \int_{\mathcal{C}_\pm} w^b e^{-w} \hat{H}(tw^\beta) \frac{\mathrm{d}w}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.

Borel transform for the heat kernel on S^2

$$\mathrm{Tr}\left(e^{(\triangle-1/(4))t}\right) \sim \frac{1}{t}\sum_{n\geq 0} B_{2n}\frac{(-1)^n(1-2^{1-2n})}{n!}t^n = \frac{1}{t}\sum_{n\geq 0} a_n t^n$$

In the previous notation, $\beta = 1$, $b = 3/2$.
The Borel transform can be summed in terms of elementary functions

$$H(\tau) = \frac{1}{\tau}\sum_{n\geq 0}\frac{a_n}{\Gamma(n+3/2)}\tau^n = \frac{1}{\sqrt{\pi\,\tau}\,\sin(\sqrt{\tau})}$$

and if we Laplace transform [Perrin, 1928]

$$s(H)(t) = \frac{2}{\sqrt{\pi}\,t^{3/2}}\int_0^\infty \mathrm{d}y\, y\frac{e^{-y^2/t}}{\sin(y)}$$

there are simple poles for $y = k\pi$, $k = 1, 2, \ldots$. The residues are

$$(2\pi i)\,\mathrm{Res}\left(\frac{2}{\sqrt{\pi}\,t^{3/2}}y\frac{e^{-y^2/t}}{\sin(y)}, k\pi\right) = (-)^{k+1}4i|k|\left(\frac{\pi}{t}\right)^{3/2}e^{-\frac{k^2\pi^2}{t}}.$$

More ingredients

Worldline interpretation

We need a non-perturbative interpretation of these exponential terms.

We read the heat kernel as the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \triangle$, *i.e.* a free quantum particle moving on $\mathbb{R} \times \Sigma$.

We can write the partition function as a path integral

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t}\int_0^1 \mathrm{d}\tau\, g_{\mu\nu}\dot{X}^\mu(\tau)\dot{X}^\nu(\tau)$$

A transseries from geodesics

In the limit $t \to 0$ the path integral localizes on a sum over all the closed geodesics γ.

For each geodesic a perturbative series in t, weighted by $e^{-\ell(\gamma)^2/(4t)}$

$$
\begin{aligned}
\operatorname{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) &= \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]} \\
&= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma \,\in\, \text{closed geodesics}}{}' e^{-\frac{\ell(\gamma)^2}{4t}} t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,
\end{aligned}
$$

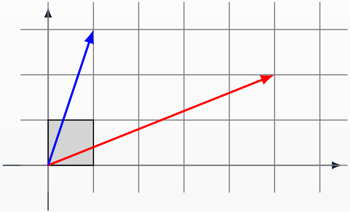
the b_γ depend on the geometry.

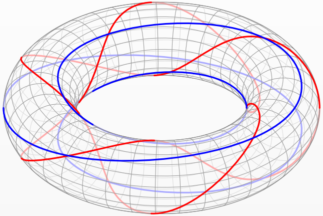
This is precisely the same structure predicted by resurgence.

Now we have a geometric interpretation.

The torus

In the case of the torus, closed geodesics are labelled by two integers (k_1, k_2)





The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h\, e^{-\frac{1}{4t}\int_0^1 d\tau\, (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t}\partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$

The torus

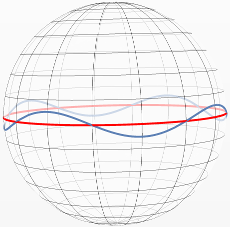
Now we can write the result of the path integral

$$\begin{aligned}\mathrm{Tr}\left(e^{\triangle t}\right) &= \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{\mathrm{cl}}} \int_{h(1)=h(0)=0} e^{-S[X_{\mathrm{cl}}]-S[h]} \\ &= \mathcal{N} L^2 \sum_{\mathbf{k}\in\mathbb{Z}^2} e^{-\frac{L^2(k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h\, e^{-S[h]}, \\ &= \frac{L^2}{4\pi t}\left[1 + \sum_{\mathbf{k}\in\mathbb{Z}^2}' e^{-\frac{L^2\|\mathbf{k}\|^2}{4t}}\right]\end{aligned}$$

This is exactly what we had found before just by looking at the spectrum.
Now we can understand the non-perturbative effects in terms of closed geodesics.

The sphere

Closed geodesics on the sphere go around the equator k times



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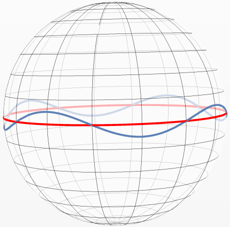

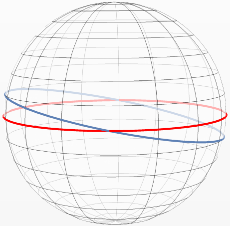

There is a zero mode because we can rotate the equator

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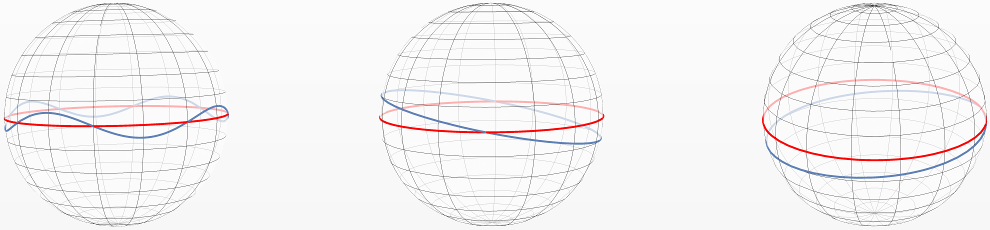

There is a zero mode because we can rotate the equator

And an instability because we can slide off

The sphere path integral

At leading order we can just pick a coordinate system and expand the action

$$L = \dot{\theta}^2 + \sin^2(\theta)\,\dot{\phi}^2$$

around the geodesic

$$\theta = \frac{\pi}{2} \qquad \phi(\tau) = 2\pi k \tau$$

so that the fluctuations give a massless and a massive mode

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k\in\mathbb{Z}} e^{-\frac{(2\pi k)^2}{4t}} \int \mathcal{D}h_\theta\, \mathcal{D}h_\phi \exp\left[-\frac{1}{4t}\int_0^1 d\tau \left(\dot{h}_\phi^2 + \dot{h}_\theta^2 - (2\pi k)^2 h_\theta^2\right)\right]$$

The sphere path integral

The h_ϕ fluctuation is massless and gives

$$\int \mathcal{D}h_\phi \exp\left[-\frac{1}{4t}\int_0^1 \mathrm{d}\tau\, \dot{h}_\phi^2\right] = \frac{1}{(4\pi t)^{1/2}}$$

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For h_θ we need to work a bit more. Decompose in modes:

$$h_\theta = \sqrt{2}\sin(\pi n \tau) \qquad \lambda_n = \frac{\pi^2}{2}\left(n^2 - 4k^2\right)$$

- a zero mode for $n = 2k$
- $2n - 1$ unstable modes

Once we regularize the determinant we get

$$\int \mathcal{D}h_\theta \exp\left[-\frac{1}{4t}\int_0^1 \mathrm{d}\tau\, \left(\dot{h}_\theta^2 - (2\pi k)^2 h_\theta^2\right)\right] = \pm i \frac{\pi}{2\sqrt{2}}\frac{k}{t}$$

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And putting it all together, the non-trivial geodesics give

$$\pm 2i\left(\frac{\pi}{t}\right)^{3/2} \sum_{k\in\mathbb{Z}}{}' |k| e^{-\frac{k^2\pi^2}{t}}$$

Back to resurgence

The one-loop result perfectly cancels the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr}\left(e^{(\triangle-\frac{1}{4})t}\right) = s_\pm(H)(t) \mp 2i\left(\frac{\pi}{t}\right)^{3/2} \sum_{k\geq 1}(-1)^k k e^{-\frac{k^2\pi^2}{t}} = \mathrm{Re}[s_\pm(H)(t)]$$

And from here we can write the exact expression for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \mathrm{Re}\left[\frac{2rm^2}{\pi}\int_0^\infty \mathrm{d}y\, \frac{K_2(2mry)}{y\sin(y)}\right] = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

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As a numerical test, we can compare with the convergent small-charge expansion ($\hat{q} \approx 0.6$)

$$r\omega(mr = 0.4)\Big|_{\text{small charge}} = 0.012\,777\,296\,63\ldots$$

$$r\omega(mr = 0.4)\Big|_{\text{resurgence}} = 0.012\,777\,297\,69\ldots$$

Optimal truncation

Lessons from large N

Let's go back to the EFT.

The effective action is identified with the asymptotic expansion: the expression we found for the grand potential is the value of the action at the minimum $\chi = \mu t$:

$$\omega(\mu) = L_{\text{EFT}}\Big|_{\chi=\mu t}$$

where

$$L_{\text{EFT}} = \omega_0(\partial_\mu \chi\, \partial^\mu \chi)^{3/2} + \omega_1(\partial_\mu \chi\, \partial^\mu \chi)^{1/2} + \dots,$$

In general the coefficients are unknown

BUT

Now we have a geometric understanding of the non-perturbative effects

Lessons from large N

Assume:

1. the large-charge expansion is asymptotic;
2. the leading pole in the Borel plane is a particle of mass μ going around the equator.

A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n \geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi \kappa f_0^{(0)} \sqrt{Q}} \sum_{n \geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \dots$$

(we used $\mu = 3 f_0^{(0)} \sqrt{Q}/2 + \dots$)

Lessons from large N

- The controlling parameter for the non-perturbative effects $e^{-3\pi\,\kappa\,f_0\sqrt{Q}}$ is fixed by the leading term in the $1/Q$ expansion.
- The non-perturbative coefficient $e^{-3\pi\,\kappa\,f_0^{(0)}\sqrt{Q}}$ fixes the large-n behavior of the perturbative series $f_n^{(0)}$.

$$f_n^{(0)} \sim (2n)!(3\pi\,\kappa\,f_0^{(0)})^{-n}$$

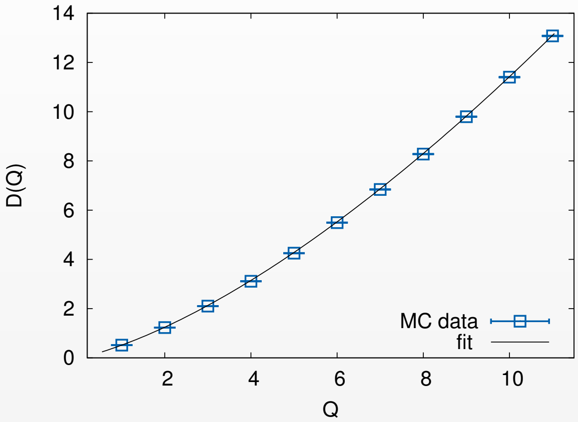
We don't know enough for a Borel resummation, but we can estimate an optimal trucation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi\,\kappa\,f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$

Can we understand the lattice results now?

In $O(2)$, $f_0^0 \approx 0.301(3)$, so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$.



This fit was obtaind with $N = 3$ terms.
For $Q = 1$ we get an error $\approx 6 \times 10^{-2}$ and for $Q = 11$ the error is $\approx 5 \times 10^{-5}$
(Compared to $e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$).

What has happened?

- The large-charge expansion of the Wilson–Fisher point is asymptotic
- In the double-scaling limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion
- We can Borel-resum the expansion
- We have a geometric interpretation for the non-perturbative effects
- We can use this geometric interpretation also in the finite-N case
- We obtain an optimal truncation and estimate of the error
- The results are consistent with lattice simulations


CONCLUSiONS
1.
2.
3.

Conclusions

- With the large-charge approach we can study strongly-coupled systems perturbatively.
- Select a sector and we write a controllable effective theory.
- The strongly-coupled physics is (for the most part) subsumed in a semiclassical state.
- Qual(nt)itative control of the non-pertubative effects.
- Compute the CFT data.
- Very good agreement with lattice (supersymmetry, large N).
- Precise and testable predictions.