Introduction to the large charge expansion

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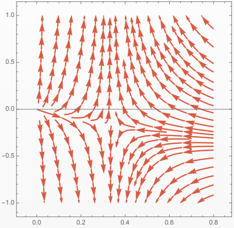
arXiv:1505.01537, arXiv:1610.04495, arXiv:1707.00711, arXiv:1804.01535, arXiv:1902.09542, arXiv:1905.00026,arXiv:1909.02571, arXiv:1909.08642, arXiv:2003.08396, arXiv:2005.03021, arXiv:2008.03308 and more to come…

Who’s who

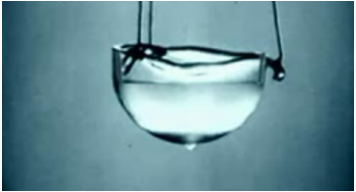
S. Reffert (AEC Bern);
L. Alvarez Gaumé (CERN and SCGP);
F. Sannino (CP3-Origins);
D. Banerjee (DESY);
S. Chandrasekharan (Duke);
S. Hellerman (IPMU);
M. Watanabe (Weizmann).

Why are we here? Conformal field theories

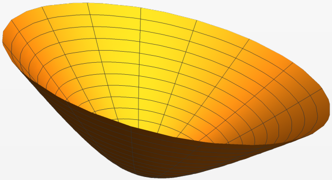
extrema of the RG flow



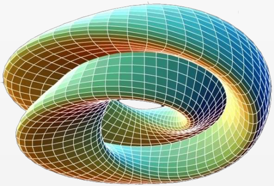
critical phenomena



quantum gravity



string theory



Why are we here? Conformal field theories are hard

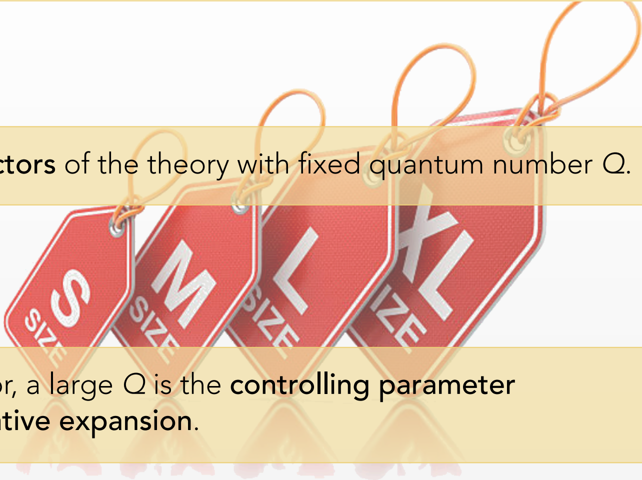
Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

Why are we here? Conformal field theories are hard

In presence of a symmetry there can be sectors of the theory where anomalous dimension and OPE coefficients simplify.

The idea



Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.

no bootstrap here!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
(However, arXiv:1710.11161).

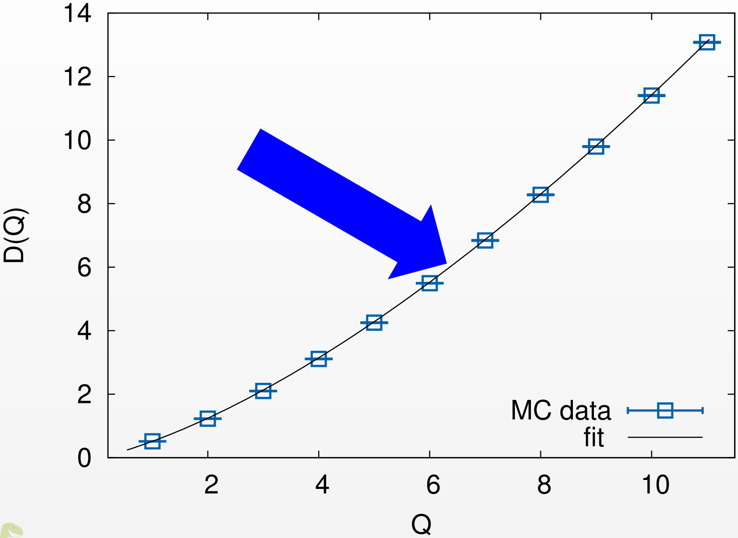
Concrete results

We consider the $O(N)$ vector model in three dimensions. In the IR it flows to a conformal fixed point Wilson & Fisher.

We find an explicit formula for the dimension of the lowest primary at fixed charge:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

Summary of the results: O(2)



Scales

We want to write a Wilsonian effective action.

Choose a cutoff Λ, separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \; e^{iS(\phi_H, \phi_L)}$$

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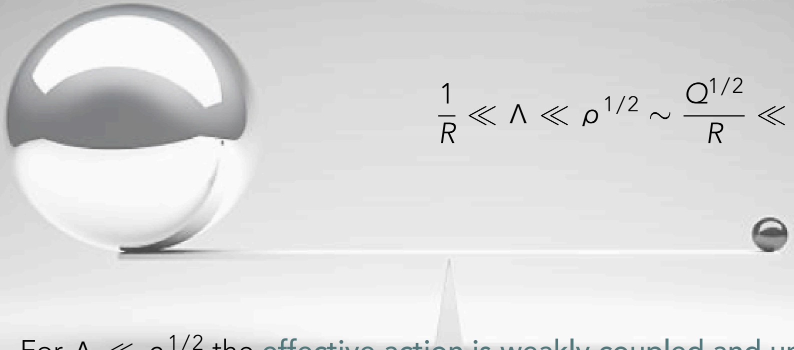
too hard

Scales

- We look at a finite box of typical length R
- The $U(1)$ charge Q fixes a second scale $\rho^{1/2} \sim Q^{1/2}/R$



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

Wilsonian action

The Wilsonian action is fundamentally useless because it contains infinite terms.

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At best:

- a cute qualitative picture;
- might allow you to get the anomalies right;
- something that helps you organize perturbative calculations, if your system is already weakly-coupled for some reason;
- *maybe* a convergent expansion in derivatives.

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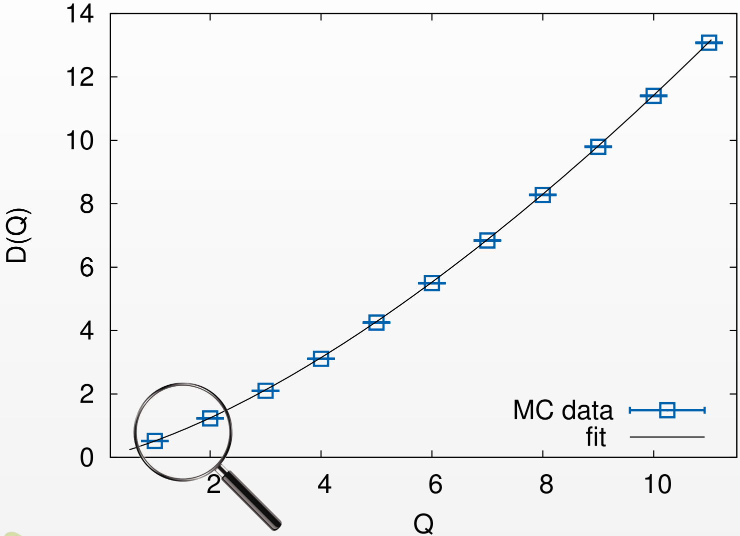
superstition

Too good to be true?

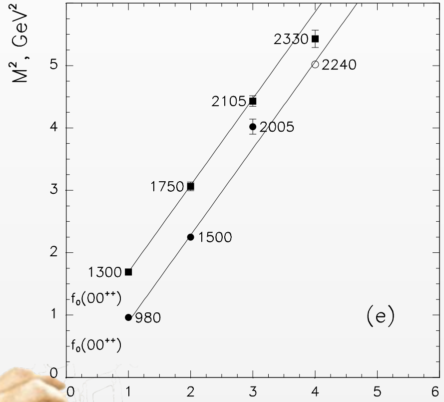


Too good to be true?

Think of Regge trajectories.
The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but *experimentally* everything works so well at small J that String Theory was invented.



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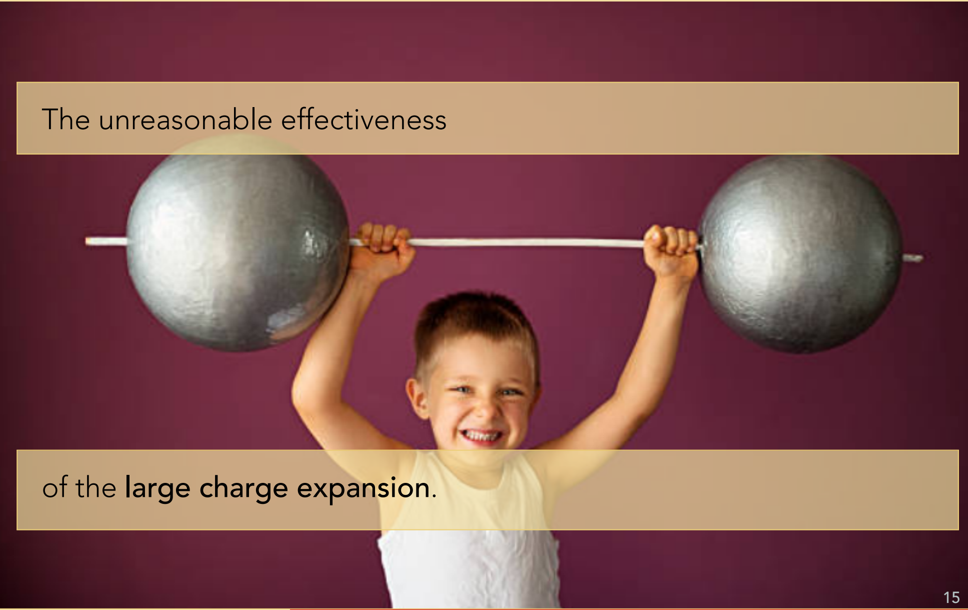

The unreasonable effectiveness

of the **large charge expansion**.

Today's talk

The EFT for the $O(2)$ model in $2+1$ dimensions

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The EFT for the $O(2)$ model in $2+1$ dimensions

- An effective field theory (EFT) for a CFT.
- The physics at the saddle.
- State/operator correspondence for anomalous dimensions.

Today's talk

The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

Today's talk

The EFT for the $O(2)$ model in $2 + 1$ dimensions

Justify and prove all my claims from first principles

Use the large-charge expansion together with supersymmetry.

- qualitatively different behavior
- compute three-point functions
- resum the large-charge expansion
- see explicitly the next saddle in the partition function

Today's talk

The EFT for the $O(2)$ model in $2+1$ dimensions

Justify and prove all my claims from first principles

Use the large-charge expansion together with supersymmetry.

Discuss some phenomenological applications

PARENTAL

ADVISORY

EXPLICIT CONTENT

An EFT for a CFT

The $O(2)$ model

The simplest example is the Wilson–Fisher (WF) point of the $O(2)$ model in three dimensions.

- Non-trivial fixed point of the ϕ^4 action

$$L_{UV} = \partial_\mu \phi^* \partial_\mu \phi - u(\phi^* \phi)^2$$

- Strongly coupled
- In nature: ^4He.
- Simplest example of spontaneous symmetry breaking.
- Not accessible in perturbation theory. Not accessible in $4 - \varepsilon$. Not accessible in large N.
- Lattice. Bootstrap.

Charge fixing

We assume that the $O(2)$ symmetry is not accidental.

We consider a subsector of fixed charge Q.
Generically, the classical solution at fixed charge breaks spontaneously $U(1) \to \varnothing$.

We have one Goldstone boson χ.

An action for χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \, \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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We want to describe a CFT: we can dress with a dilaton

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \, \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \, \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken $U(1)$, the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

Linear sigma model

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

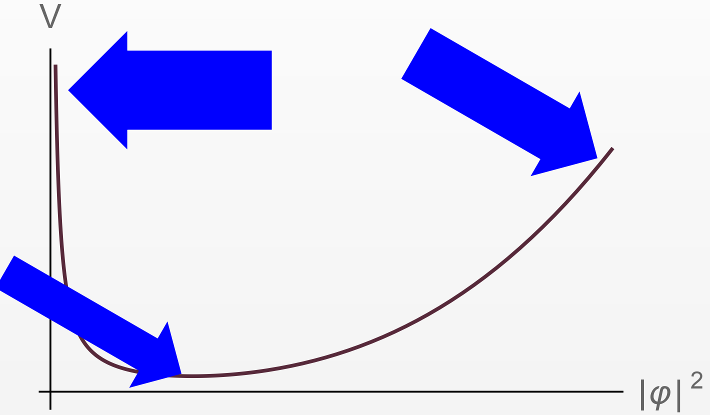
We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.
Scale invariance is manifest.
The field φ is some complicated function of the original ϕ.

Centrifugal barrier

The $O(2)$ symmetry acts as a shift on χ.
Fixing the charge is the same as adding a centrifugal term $\propto \frac{1}{|\varphi|^2}$.



Ground state

We can find a fixed-charge solution of the type

$$\chi(t,x) = \mu t \qquad \sigma(t,x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \dots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2} V Q^{3/2} + c_{1/2} R V Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

Fluctuations

The fluctuations over this ground state are described by two modes.

- A universal "conformal Goldstone". It comes from the breaking of the $U(1)$.

$$\omega = \frac{1}{\sqrt{2}} p$$

- The massive dilaton. It controls the magnitude of the quantum fluctuations. All quantum effects are controled by $1/Q$.

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

Non-linear sigma model

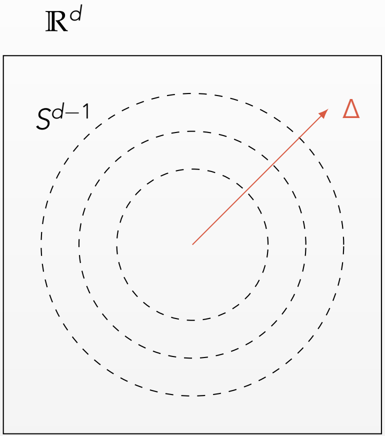
Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

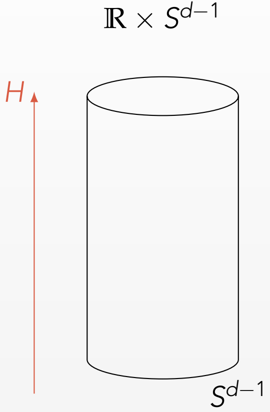
$$L[\chi] = k_{3/2}(\partial_\mu \chi \, \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \, \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$. All other terms are suppressed by powers of $1/Q$.

State-operator correspondence

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.





Protected by conformal invariance: a well-defined quantity.

Conformal dimensions

We know the energy of the ground state.
The leading quantum effect is the Casimir energy of the conformal Goldstone.

$$E_G = \frac{1}{2\sqrt{2}} \zeta\left(-\tfrac{1}{2} | S^2\right) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the conformal dimension of the lowest operator of charge Q in the $O(2)$ model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

The $O(2N)$ model

Next step: $O(2N)$. We take $2N$ fields and an action that is invariant under

$$\phi^a \to M^a_b \phi^b, \qquad M^T M = \mathbb{1}.$$

The conserved current associated to the global $O(2n)$ symmetry is now matrix-valued and has the form

$$(j^\mu)^{ab} = (\phi^a \partial^\mu \phi^b - \phi^b \partial^\mu \phi^a).$$

we can only fix the $\mathrm{rank}(O(2n))$ coefficients in the directions of the mutually commuting Cartan generators H^I.

$$q_I = \frac{1}{2}\langle Q H^I \rangle, \qquad \left[H^I, H^J\right] = 0, \qquad \langle H^I H^J \rangle = 2\delta^{IJ}.$$

The $O(2N)$ model

The q_I transform under the action of $O(2n)$, while the spectrum of the system is invariant.
The energy of a state of fixed charge Q can only depend on the conjugacy class of Q.
There exists a homogeneous ground state. There is always an $O(2n)$ transformation M such that

$$MQM^{-1} = \sum_{I=1}^{n} \hat{q}_I H^I = \begin{pmatrix} 0 & \hat{q} & & & \\ -\hat{q} & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ & & & & \ddots \end{pmatrix} .$$

where

$$\hat{q} = q_1 + \dots q_N$$

The $O(2N)$ model

The ground-state energy only depends on the sum of the charges

$$\hat{q} = q_1 + \cdots + q_N$$

and takes the same form

$$E = \frac{c_{3/2}(N)}{2\sqrt{\pi}} \hat{q}^{3/2} + 2\sqrt{\pi}\, c_{1/2}(N) \hat{q}^{1/2} + \mathcal{O}\left(\hat{q}^{-1/2}\right)$$

The coefficients depend on N and cannot be computed in the EFT (but *e.g.* in large-N).

Fluctuations

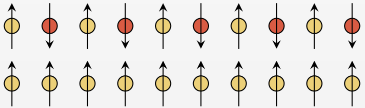
The symmetry breaking pattern is

$$O(2N) \xrightarrow{\text{exp.}} U(N) \xrightarrow{\text{spont.}} U(N-1)$$

and there are $\dim(U(N)/U(N-1)) = 2N-1$ degrees of freedom (DOF).

- One singlet, the universal conformal Goldstone $\omega = \frac{1}{\sqrt{2}} p$
- One vector of $U(N-1)$, with quadratic dispersion $\omega = \frac{p^2}{2\mu} + \dots$

⚠ We have singled out the time. The system is non-relativistic.



antiferromagnet $\omega \propto p$

ferromagnet $\omega \propto p^2$ (count double)

Type II Goldstones

The inverse propagator for the type-II is

$$D^{-1} = \begin{pmatrix} \frac{1}{2}(\nabla^2 - \partial_0^2) & \mu\,\partial_0 \\ -\mu\,\partial_0 & \frac{1}{2}(\nabla^2 - \partial_0^2) \end{pmatrix}$$

and the dispersion relation

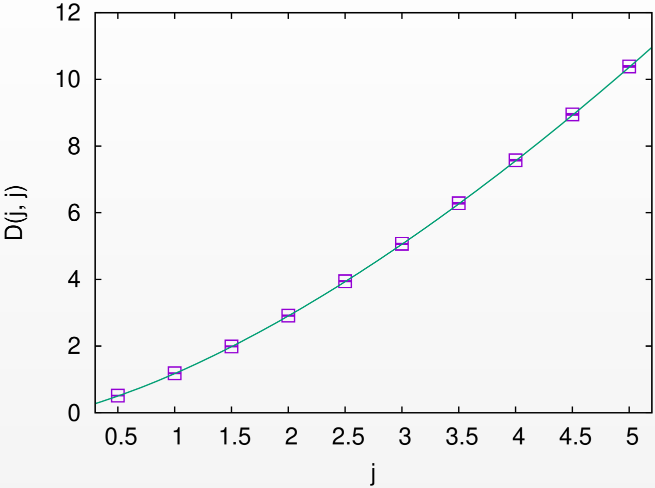
$$\omega = \sqrt{p^2 + \mu^2} \pm \mu\,.$$

Each type-II Goldstone counts for two DOF:

$$1 + 2 \times (N - 1) = 2N - 1.$$

Only the type-I has a Q^0 contribution: it is universal.

$O(4)$ on the lattice



$$\Delta_j = \frac{c_{3/2}}{2\sqrt{\pi}}(2j)^{3/2} + 2\sqrt{\pi}\, c_{1/2}(2j)^{1/2} - 0.094 + \mathcal{O}\left(j^{-1/2}\right)$$

What happened?

We started from a CFT.
There is no mass gap, there are no particles, there is no Lagrangian.

We picked a sector.

In this sector the physics is described by a semiclassical configuration plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a simple EFT.
We are in a strongly coupled regime but we can compute physical observables using perturbation theory.

Large N *vs.* Large Charge

The model

ϕ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int \mathrm{d}t \, \mathrm{d}\Sigma \left[g^{\mu\nu} (\partial_\mu \varphi_i)^* (\partial_\nu \varphi_i) + r \varphi_i^* \varphi_i + \frac{u}{2} (\varphi_i^* \varphi_i)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.
We compute the partition function at fixed charge

$$Z(Q_1, \dots, Q_N) = \operatorname{Tr} \left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int \mathrm{d}\Sigma \, j_i^0 = i \int \mathrm{d}\Sigma \, [\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i].$$

Dimensions of operators of fixed charge Q on $\mathbb{R}^3$ (state/operator):

$$\Delta(Q) = -\frac{1}{\beta} \log Z_{S^2}(Q).$$

Fix the charge

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i} \,\mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$\begin{aligned} Z_\Sigma(Q) &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i \, e^{-S[\varphi]} \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i \, e^{-S^\theta[\varphi]} \end{aligned}$$

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Effective actions

The covariant derivative approach:

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u(\varphi_i^* \varphi_i)^2 \right)$$

where

$$\begin{cases} D_0 \varphi &= \partial_0 \varphi + i\frac{\theta}{\beta} \varphi \\ D_i \varphi &= \partial_i \varphi \end{cases}$$

Stratonovich transformation: introduce Lagrange multiplier λ and rewrite the action as

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int \mathrm{d}t\,\mathrm{d}\Sigma \left[\left(D^i_\mu \varphi_i \right)^* \left(D^i_\mu \varphi_i \right) + (r + \lambda)\varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = \left(m^2 - r \right) + \hat{\lambda}$$

Saddle point equations

With some massaging, we find the final equations

$$\begin{cases} F_{\Sigma}^{\bigcirc}(Q) = mQ + N\zeta\left(-\frac{1}{2}\middle|\Sigma, m\right), \\ m\zeta\left(\frac{1}{2}\middle|\Sigma, m\right) = -\frac{Q}{N}. \end{cases}$$

The control parameter is actually Q/N.

Large Q/N

If $Q/N \gg 1$ we can use Weyl's asymptotic expansion.

$$\mathrm{Tr}\left(e^{\Delta_\Sigma t}\right) = \sum_{n=0}^{\infty} K_n t^{n/2-1}.$$

The zeta function is written in terms of the geometry of Σ (heat kernel coefficients)

$$m_\Sigma = \sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{1/2} + \frac{R}{24}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{-1/2} + \dots$$

$$\frac{F_\Sigma}{2N} = \frac{2}{3}\sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{3/2} + \frac{R}{12}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{1/2} + \dots$$

Order N

$$
\begin{aligned}
F_{S^2}(Q) = {} & \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} \\
& - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)
\end{aligned}
$$

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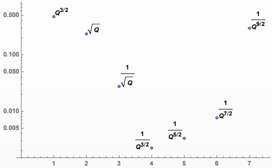

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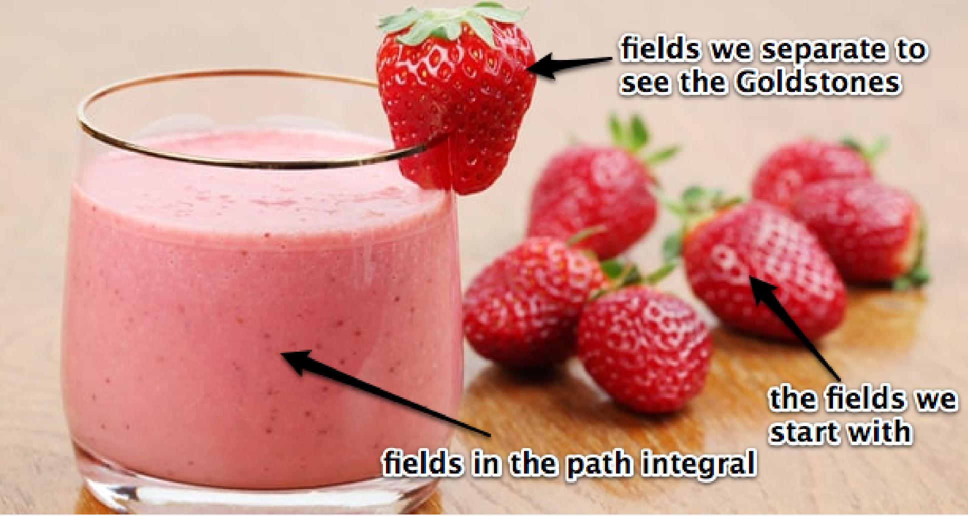
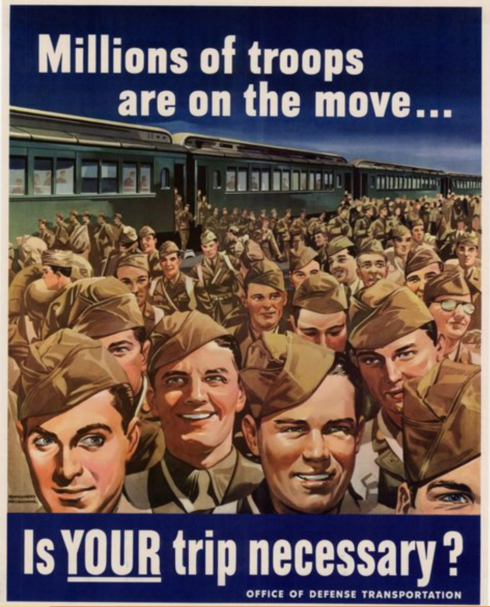
Where is the universal Goldstone?



Was it worth it?



Final result

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{1/2} + \dots$$
$$- 0.0937\dots$$

would you like to know more?

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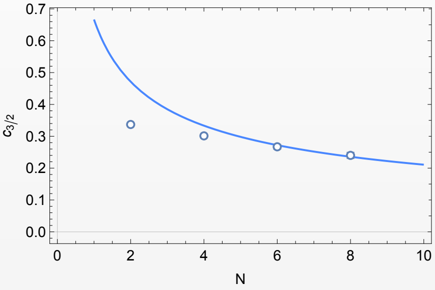

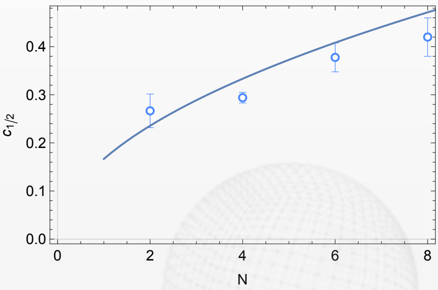

▸▸ would you like to know more?

Large charge and supersymmetry

And Now for Something Completely Different

All the models that you have seen have something in common: isolated vacuum. No moduli space.
What happens when there is a flat direction?

Many known examples of (non-Lagrangian) $\mathcal{N} \geq 2$ SCFT in four dimensions.
Coulomb branch with a dimension-one moduli space: all the physics is encoded in a single operator $\mathcal{O}$ and every chiral operator is just $\mathcal{O}^n$.

We will write an effective action for a canonically-normalized dimension-one vector multiplet Φ.

Effective action

We have a single vector multiplet. The kinetic term is just

$$L_k = \int \mathrm{d}^4\theta \ \Phi^2 + \text{c.c.} = |\partial \phi|^2 + \text{fermions} + \text{gauge fields}$$

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$$L^{\text{EFT}} = L_K + \alpha L_{WZ}$$

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$$L^{\text{EFT}} = L_K + \alpha L_{WZ}$$

The coefficient α fixes the a-anomaly of the EFT. It has to match the anomaly in the UV.
Claim: at large R-charge this action is all you need for any $\mathcal{N} = 2$ theory (with one-dimensional moduli space).

Observables

Three-point function of the Coulomb branch operators

$$\left\langle \Phi^{n_1}(x_1)\Phi^{n_2}(x_2)\bar{\Phi}^{n_1+n_2}(x_3)\right\rangle = \frac{C^{n_1,n_2,n_1+n_2}}{|x_1-x_3|^{2n_1\Delta}|x_2-x_3|^{2n_2\Delta}}$$

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The OPE of Φ with itself is regular, so we can set $x_2 = x_1$ and the three-point function is actually a two-point function.

$$C^{n',n-n',n} = |x_1-x_2|^{2n\Delta}\left\langle \Phi^{n}(x_1)\bar{\Phi}^{n}(x_2)\right\rangle$$

$Q = n\Delta$ is the controlling parameter (it's the R-charge)

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The coefficients satisfy a Toda lattice equation that can be solved using as boundary condition the one loop EFT computation.

Final result

The final result for the generator $\mathcal{O}$ of the Coulomb branch is:

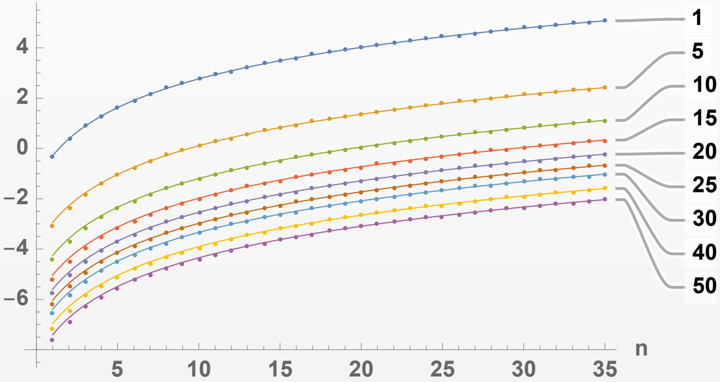
$$\langle \mathcal{O}^n(x_1)\bar{\mathcal{O}}^n(x_2)\rangle = C_n(\tau, \bar{\tau}) \frac{\Gamma(2n\Delta + \alpha + 1)}{|x_1 - x_2|^{2n\Delta}}$$

The coefficient C_n is scheme-dependent.
The gamma term is universal, only depends on α.
This result is valid for any rank-one theory, Lagrangian or not.

We have completely resummed the $1/Q$ expansion.

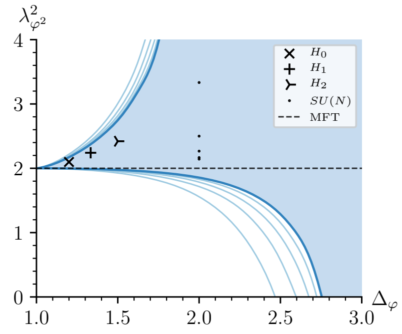
Comparison with localization

How well does this work?
For the special case of $SU(2)$ SQCD with $N_f = 4$ we can compare with localization. arXiv:1602.05971



Comparison with boostrap

For strongly coupled theories one can use bootstrap to place bounds on the three-point coefficients with $n = 1$.
This is the worst possible situation for us. And still...



Taken from arXiv:2006.01847

would you like to know more?

An asymptotically safe QFT

IR vs. UV

We have discussed an IR fixed point.
The fixed charge induces a scale $\Lambda_Q = \frac{Q^{1/d}}{r}$.
We need a hierarchy for the scale Λ of the EFT

$$\frac{1}{r} \ll \Lambda \ll \Lambda_Q \ll \Lambda_{UV}$$

The situation improves if we consider a ultraviolet (UV) fixed point.

$$\frac{1}{r} \ll \Lambda_{UV} \ll \Lambda \ll \Lambda_Q$$

and we can take the charge as large as we like.

An asymptotically safe theory

$$\mathcal{L} = -\frac{1}{2}\operatorname{Tr}(F^{\mu\nu}F_{\mu\nu}) + \operatorname{Tr}\left(\bar{Q} i \not{D} Q\right) + y \operatorname{Tr}\left(\bar{Q}_L H Q_R + \bar{Q}_R H^\dagger Q_L\right)$$
$$+ \operatorname{Tr}\left(\partial_\mu H^\dagger \partial^\mu H\right) - u \operatorname{Tr}\left(H^\dagger H\right)^2 - v(\operatorname{Tr} H^\dagger H)^2 - \frac{R}{6}\operatorname{Tr}\left(H^\dagger H\right).$$

In the Veneziano limit of $N_F \to \infty$, $N_C \to \infty$ with the ratio N_F/N_C fixed, this theory is asymptotically safe.

Perturbatively-controlled UV fixed point

$$\alpha_g^* = \frac{26}{57}\varepsilon\,, \quad \alpha_y^* = \frac{4}{19}\varepsilon\,, \quad \alpha_h^* = \frac{\sqrt{23}-1}{19}\varepsilon\,, \quad \alpha_v^* = -0.13\,\varepsilon\,.$$

An asymptotically safe theory

New features from our point of view

- H is a matrix. There is a large non-Abelian global symmetry
- there are fermions
- there are gluons
- it's a four-dimensional system
- we have a trustable effective action

The scalar sector

Inspired by the $O(2)$ model we use a homogeneous ansatz

$$H_0 = e^{2iMt} B,$$

and the equations of motion (EOM) reduce to

$$2M^2 = uB^2 + v \operatorname{Tr}\left(B^2\right) - \frac{R}{12}.$$

For simplicity

$$\mathcal{Q}_L = -\mathcal{Q}_R = J \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right),$$

where $\mathbb{1}$ is the $N_F/2 \times N_F/2$ identity matrix.
The ground state is

$$M = \mu \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right), \qquad B = b \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & \mathbb{1} \end{array}\right).$$

Ground state energy and fluctuations

The ground state has energy

$$E = \frac{3}{2}\frac{N_F^2}{\alpha_h + \alpha_v}\left(\frac{2\pi}{V}\right)^{1/3}\left[\mathcal{J}^{4/3} + \frac{R}{36}\left(\frac{V}{2\pi^2}\right)^{2/3}\mathcal{J}^{2/3} - \frac{1}{144}\left(\frac{R}{6}\right)^2\left(\frac{V}{2\pi^2}\right)^{4/3}\mathcal{J}^0 + \mathcal{O}\left(\mathcal{J}^{-2/3}\right)\right]$$

which is a natural expansion in

$$\mathcal{J} = 2J\frac{\alpha_h + \alpha_v}{N_F} \gg 1$$

We have again an expansion in powers of the charge.
The leading exponent is $4/3$ because we are in four dimensions.

Goldstones

The symmetry-breaking pattern is quite involved

$$SU(N_F) \times SU(N_F) \times U(1) \xrightarrow{\text{exp.}} C(M) \times SU(N_F) \xrightarrow{\text{spont.}} C(M).$$

where $C(M) = SU(N_F/2) \times SU(N_F/2) \times U(1)^2$.

Type-I and type-II Goldstones.

- One conformal Goldstone $\omega = \frac{p}{\sqrt{3}}$, which is a singlet of $C(M)$
- One bifundamental with $\omega = \frac{p^2}{2\mu}$
- One field in the $(\mathbf{Adj}, \mathbf{1})$ and one in the $(\mathbf{1}, \mathbf{Adj})$ with $\omega = \sqrt{\frac{\alpha_h}{3\alpha_h + 2\alpha_v}} p$

Total count:

$$1 + 2 \times (N_F/2)^2 + 2 \times (N_F^2/4 - 1) = N_F^2 - 1 = \dim(SU(N_F))$$

Summing it up

- We can use the large-charge expansion for asymptotically safe theories
- Being in the UV, the large-charge condition is more natural
- For the QCD-inspired model that we have considered:
  - Fermions and gluons decouple.
  - $1/\mathcal{J}$ expansion of the anomalous dimensions, starting at $\mathcal{J}^{4/3}$
  - Rich spectrum of Goldstone modes, with linear and quadratic dispersions.

would you like to know more?

In conclusion

- With the large-charge approach we can study strongly-coupled systems perturbatively.
- Select a sector and we write a controllable effective theory.
- The strongly-coupled physics is (for the most part) subsumed in a semiclassical state.
- Compute the CFT data.
- Very good agreement with lattice (supersymmetry, large N).
- Precise and testable predictions.

Large N *vs.* Large Charge

The model

ϕ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int \mathrm{d}t\, \mathrm{d}\Sigma \left[g^{\mu\nu} (\partial_\mu \varphi_i)^* (\partial_\nu \varphi_i) + r \varphi_i^* \varphi_i + \frac{u}{2} (\varphi_i^* \varphi_i)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.
We compute the partition function at fixed charge

$$Z(Q_1, \dots, Q_N) = \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int \mathrm{d}\Sigma\, j_i^0 = i \int \mathrm{d}\Sigma\, [\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i].$$

Dimensions of operators of fixed charge Q on $\mathbb{R}^3$ (state/operator):

$$\Delta(Q) = -\frac{1}{\beta} \log Z_{S^2}(Q).$$

Fix the charge

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i} \operatorname{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i \, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i \, e^{-S^\theta[\varphi]}$$

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Effective actions

The covariant derivative approach:

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u (\varphi_i^* \varphi_i)^2 \right)$$

where

$$\begin{cases} D_0 \varphi &= \partial_0 \varphi + i \frac{\theta}{\beta} \varphi \\ D_i \varphi &= \partial_i \varphi \end{cases}$$

Stratonovich transformation: introduce Lagrange multiplier λ and rewrite the action as

$$S_Q = \sum_{i=1}^{N} \left[-i \theta_i Q_i + \int \mathrm{d}t\,\mathrm{d}\Sigma \left[\left(D^i_\mu \varphi_i \right)^* \left(D^i_\mu \varphi_i \right) + (r + \lambda) \varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = \left(m^2 - r \right) + \hat{\lambda}$$

Effective action for $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_\theta[\hat{\lambda}] = \sum_{i=1}^{N} \left[V\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \operatorname{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 + \hat{\lambda} \right) \right] - \frac{A_i^2}{2} \operatorname{Tr}\left(\hat{\lambda}\, \Delta\, \hat{\lambda} \right) \right].$$

This is a non-local action for $\hat{\lambda}$, that can be expanded order-by-order in $1/N$. Today we will only look at the leading order (saddle point).

Saddle point equations

$$
\begin{cases}
\dfrac{\partial S_Q}{\partial m^2} = \displaystyle\sum_{i=1}^{N} \left[\frac{V\beta}{2} A_i^2 + \zeta(1 | \theta_i, \Sigma, m) \right] = 0, \\
\dfrac{\partial S_Q}{\partial \theta_i} = -iQ + \dfrac{\theta_i}{\beta} V A_i^2 + \dfrac{1}{s} \dfrac{\partial}{\partial \theta_i} \zeta(s | \theta_i, \Sigma, m) \Big|_{s=0} = 0 \\
\dfrac{\partial S_Q}{\partial A_i} = V\beta \left(\dfrac{\theta_i^2}{\beta^2} + m^2 \right) A_i = 0.
\end{cases}
$$

where

$$
\zeta(s | \theta, \Sigma, m) = \sum_{n \in \mathbb{Z}} \sum_{p} \left(\left(\frac{2\pi n}{\beta} + \frac{\theta}{\beta} \right)^2 + E(p)^2 + m^2 \right)^{-s}.
$$

Saddle point equations

With some massaging, we find the final equations

$$\begin{cases} F_{\Sigma}^{\circledast}(Q) = mQ + N\zeta\left(-\frac{1}{2}\middle|\Sigma, m\right), \\ m\zeta\left(\frac{1}{2}\middle|\Sigma, m\right) = -\frac{Q}{N}. \end{cases}$$

The control parameter is actually Q/N.

Small Q/N

The zeta function can be expanded in perturbatively in small Q/N. Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2}\frac{Q}{N} + \frac{16\left(\pi^2 - 12\right) Q^2}{3\pi^4 N^2} + \dots$$

- Expansion of a closed expression
- Start with the engineering dimension $1/2$
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation

Large Q/N

If $Q/N \gg 1$ we can use Weyl's asymptotic expansion.

$$\operatorname{Tr}\left(e^{\Delta_\Sigma t}\right) = \sum_{n=0}^{\infty} K_n t^{n/2-1}.$$

The zeta function is written in terms of the geometry of Σ (heat kernel coefficients)

$$m_\Sigma = \sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{1/2} + \frac{R}{24}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{-1/2} + \dots$$

$$\frac{F_\Sigma}{2N} = \frac{2}{3}\sqrt{\frac{4\pi}{V}}\left(\frac{Q}{2N}\right)^{3/2} + \frac{R}{12}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{1/2} + \dots$$

Order N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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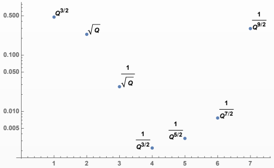

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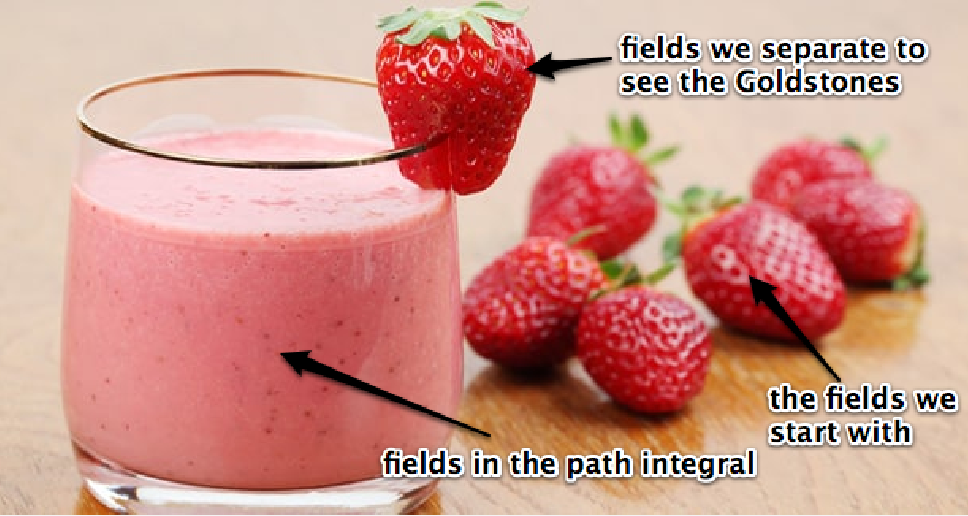
Universal term: integrate all but one



Order N^0

The order N^0 terms are

$$S^\theta[\hat\sigma,\hat\lambda] = \int \mathrm{d}t\,\mathrm{d}\Sigma\,\Big((D_\mu\hat\sigma)^*(D^\mu\hat\sigma) + (m^2+\hat\lambda)\hat\sigma^*\hat\sigma + \frac{\hat\lambda v(\hat\sigma+\hat\sigma^*)}{(N-1)^{1/2}}\Big) + \frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2\,\hat\lambda(x_1)\hat\lambda(x_2)D(x_1-x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.
At low energies we can approximate the non-local term as

$$\int \mathrm{d}t\,\mathrm{d}\Sigma\,\hat\lambda(x)^2\zeta(2|\theta,\Sigma,m) \approx \frac{V}{2m}\int \mathrm{d}t\,\mathrm{d}\Sigma\,\hat\lambda(x)^2$$

and we can integrate $\hat\lambda$ out.

Order N^0

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4m^2) & m\omega \\ -m\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \dots = 0 \qquad \omega^2 + 8m^2 + \frac{3}{2}p^2 + \dots = 0$$

This is the conformal Goldstone that we have seen in the EFT.
Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\dots$$

This is universal. Does not depend on N or Q.

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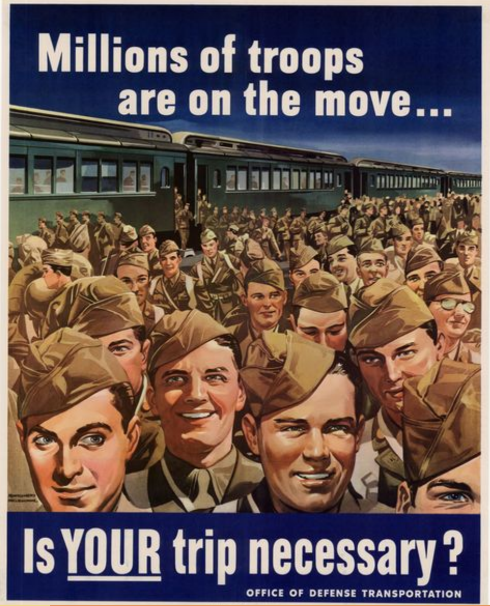
Was it worth it?



Final result

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{1/2} + \dots$$
$$- 0.0937\dots$$

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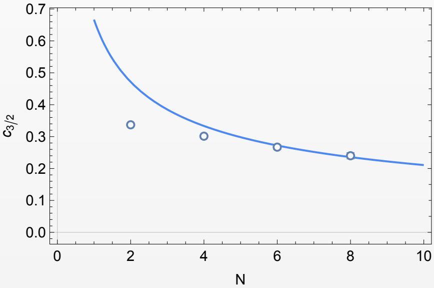

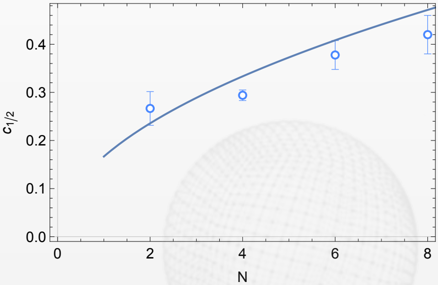

Large charge and supersymmetry

And Now for Something Completely Different

All the models that you have seen have something in common: isolated vacuum. No moduli space.
What happens when there is a flat direction?

Many known examples of (non-Lagrangian) $\mathcal{N} \geq 2$ SCFT in four dimensions.
Coulomb branch with a dimension-one moduli space: all the physics is encoded in a single operator $\mathcal{O}$ and every chiral operator is just $\mathcal{O}^n$.
We will write an effective action for a canonically-normalized dimension-one vector multiplet Φ.

Effective action

We have a single vector multiplet. The kinetic term is just

$$L_k = \int \mathrm{d}^4\theta \;\Phi^2 + \text{c.c.} = |\partial\phi|^2 + \text{fermions} + \text{gauge fields}$$

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$$L^{\text{EFT}} = L_K + \alpha L_{WZ}$$

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$$L^{\text{EFT}} = L_K + \alpha L_{WZ}$$

The coefficient α fixes the a-anomaly of the EFT. It has to match the anomaly in the UV.

Claim: at large R-charge this action is all you need for any $\mathcal{N} = 2$ theory (with one-dimensional moduli space).

Observables

Three-point function of the Coulomb branch operators

$$\left\langle \Phi^{n_1}(x_1)\Phi^{n_2}(x_2)\bar{\Phi}^{n_1+n_2}(x_3)\right\rangle = \frac{C^{n_1,n_2,n_1+n_2}}{|x_1-x_3|^{2n_1 D}|x_2-x_3|^{2n_2 D}}$$

The OPE of Φ with itself is regular, so we can set $x_2 = x_1$ and the three-point function is actually a two-point function.

$$C^{n',n-n',n} = |x_1-x_2|^{2nD}\left\langle \Phi^{n}(x_1)\bar{\Phi}^{n}(x_2)\right\rangle = e^{q_n-q_0}$$

$Q = nD$ is the controlling parameter (it's the R-charge)

Two-point function

$$\left\langle \Phi^n(x_1)\bar{\Phi}^n(x_2)\right\rangle = \int D\phi \; \phi^n(x_1)\,\bar{\phi}^n(x_2) e^{-S_k}$$

We can just pull the sources in the action and minimize

$$S_k + S_{\text{sources}} \propto k_0 + \int \mathrm{d}^4x \Big[\partial_\mu \phi \, \partial_\mu \bar{\phi} \\ - Q \log \phi \, \delta\left(x - x_1\right) - Q \log \bar{\phi}\, \delta\left(x - x_2\right)\Big]$$

At the minimum:

$$S = k_0 + k_1 Q - Q \log Q + 2Q \log |x_1 - x_2| + \mathcal{O}\left(Q^0\right)$$

so

$$q_n = k_0 + k_1 Q + \left(Q + \tfrac{1}{2}\right) \log(Q) + \mathcal{O}\left(Q^0\right)$$

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Two-point function: tree level

Adding the WZ term gives another contribution

$$q_n = k_1 Q + k_0 + Q \log(Q) + \left(\alpha + \tfrac{1}{2}\right) \log(Q) + \mathcal{O}\left(Q^0\right)$$

This is the tree-level result.

Corrections from quantum fluctuations in the path integral.
No other tree-level terms.

Two-point function: quantum corrections

$1/Q$ is the loop-counting parameter because we are expanding around a vacuum expectation value (VEV) that depends on Q.
Sum of a ground state piece and a series in $1/Q$.

$$q_n = k_0 + k_1 Q + Q \log(Q) + \left(\alpha + \tfrac{1}{2}\right) \log(Q) + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

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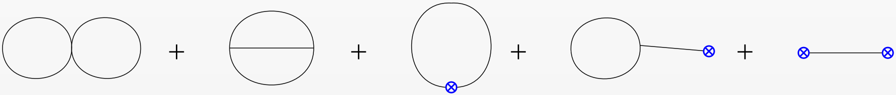
Compute order-by-order



$$k_1(\alpha) = \frac{1}{2}\left(\alpha^2 + \alpha + \frac{1}{6}\right)$$

Supersymmetry to the rescue

There is a better way.
The q_n satisfy a Toda lattice equation arXiv:0910.4963

$$\partial_\tau \partial_{\bar{\tau}} q_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$$

This is integrable, but it's hard to find explicit solutions.
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$$q_n = k_0(\tau, \bar\tau) + Qf(\tau, \bar\tau) + Q\log(Q) + \left(\alpha + \tfrac{1}{2}\right)\log(Q) + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

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Recursion relation

We can actually solve the recursion relation, using the value of $k_1(\alpha)$ found at one loop.

$$q_n = k_0(\tau, \bar{\tau}) + Qf(\tau, \bar{\tau}) + \log(\Gamma(2n + \alpha + 1))$$

The log term is universal, only depends on α.

We have completely resummed the $1/Q$ expansion.

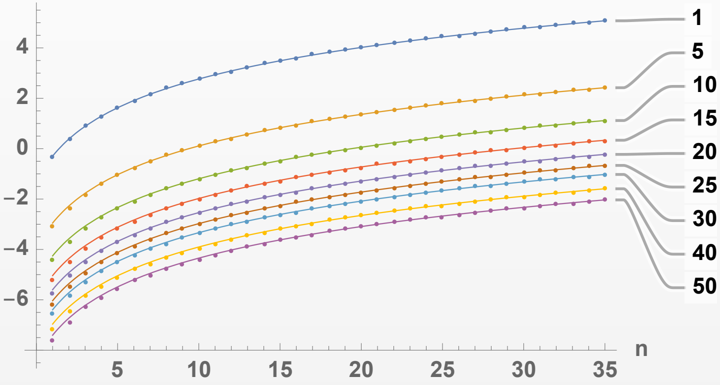
In terms of the generator $\mathcal{O}$ of the Coulomb branch we have:

$$\langle \mathcal{O}^n(x_1) \bar{\mathcal{O}}^n(x_2) \rangle = C_n(\tau, \bar{\tau}) \frac{\Gamma(2n\Delta + \alpha + 1)}{|x_1 - x_2|^{2n\Delta}}$$

The coefficient C_n is scheme-dependent.

Comparison with localization

How well does this work?
For the special case of $SU(2)$ SQCD with $N_f = 4$ we can compare with localization. arXiv:1602.05971



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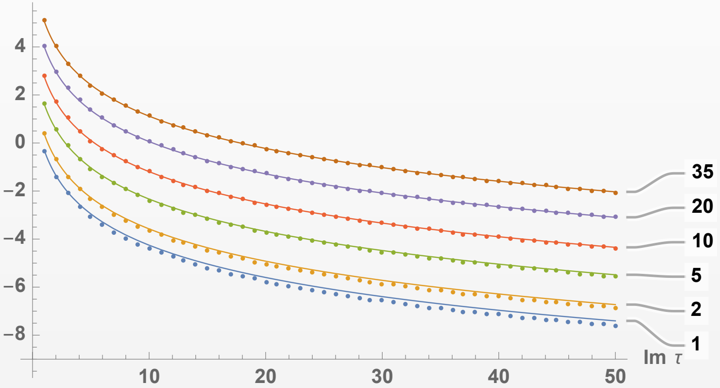
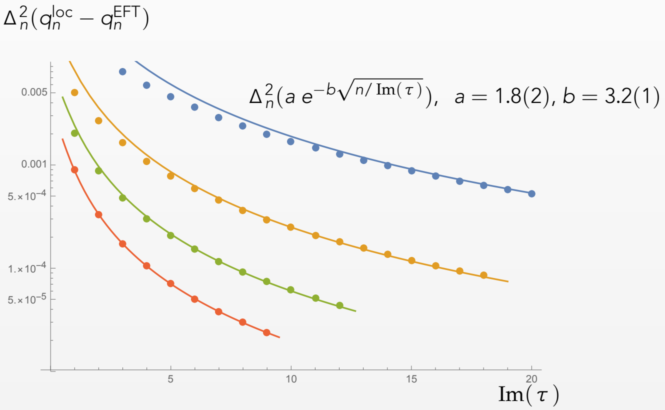
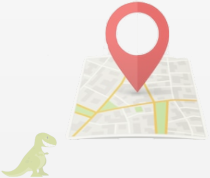


A semi-empirical instanton

A semi-empirical instanton

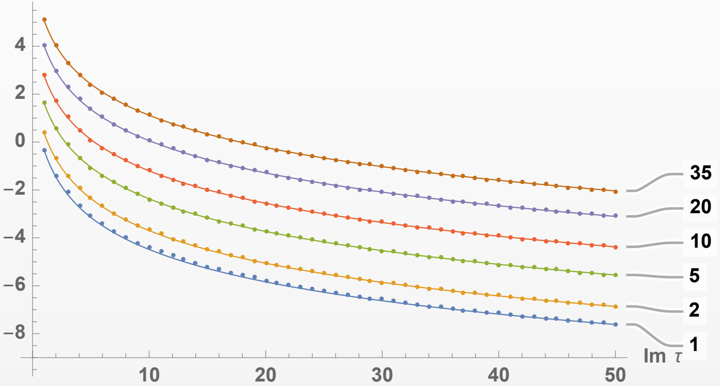
We can do better.
We have resummed the $1/Q$ expansion around one vacuum.
Exponential corrections coming from the next saddle in the path integral.

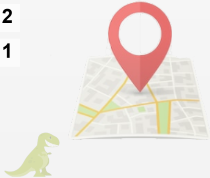




Comparison with localization

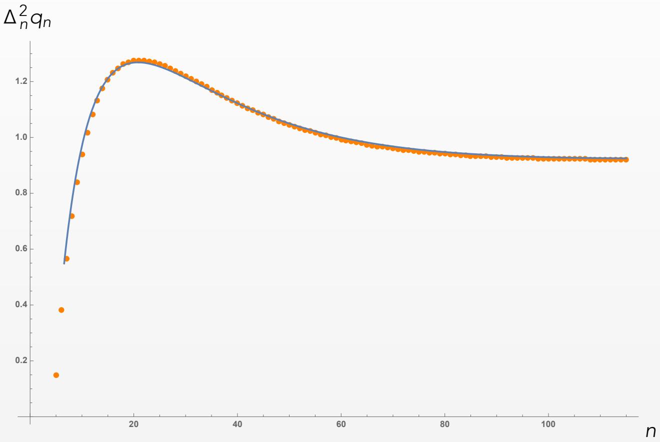
Once we add the first exponential correction

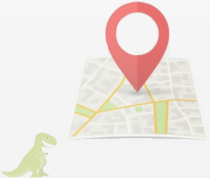




Comparison with localization

Once we add the first exponential correction (fixed $\tau = 6$)

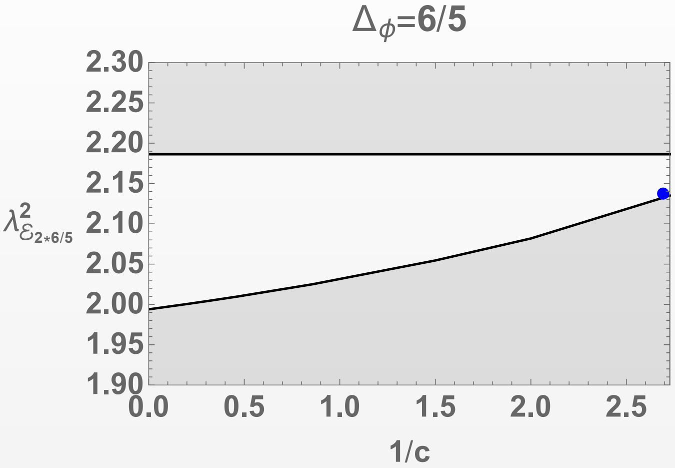




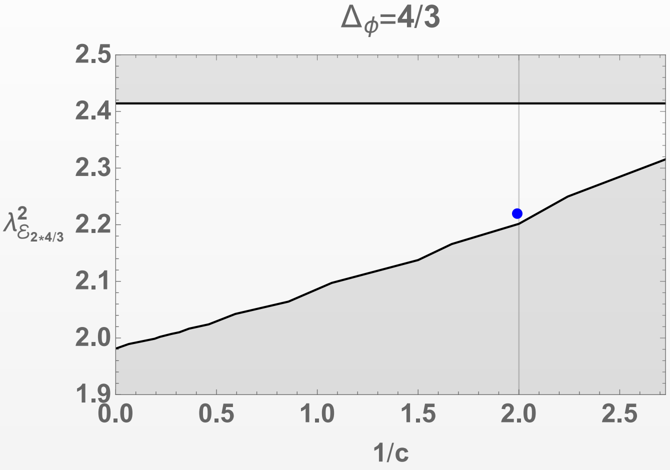
Comparison with bootstrap

For strongly coupled theories one can use bootstrap to place bounds on the three-point coefficients with $n = 1$.
This is the worst possible situation for us. And still…

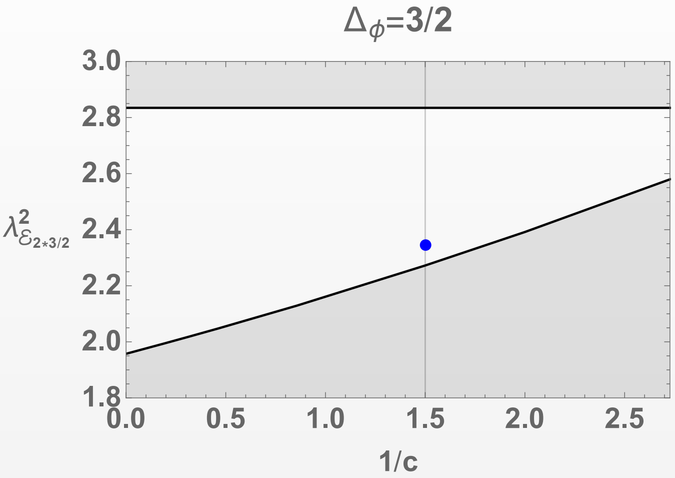
Comparison with bootstrap



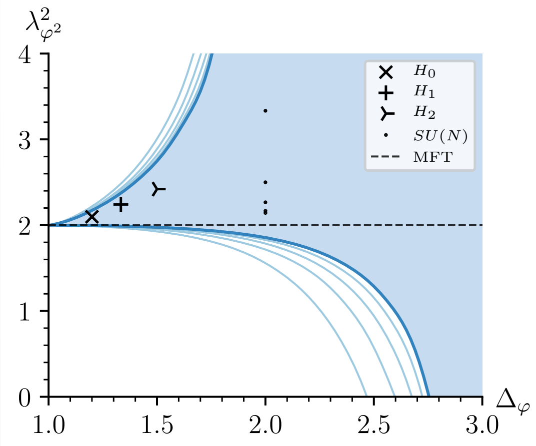
Comparison with bootstrap



Comparison with bootstrap



Comparison with boostrap



Taken from arXiv:2006.01847

Going away from conformality

Going away away from conformality

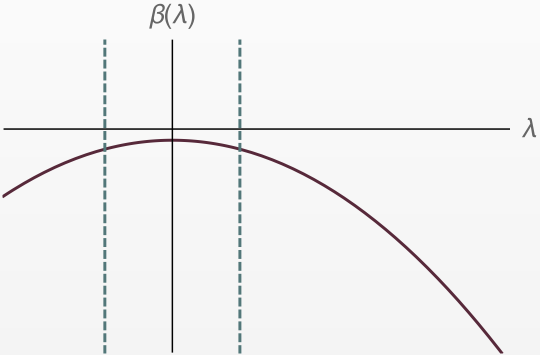
CFTs are very interesting but very constrained.

There is a lot of interesting physics that happens away from conformality.
If we don't go "too far" we can still use large charge effectively.

We will find a very distinct signature of new physics associated to a small dilaton mass in the EFT.

Walking dynamics

For example the walking phase when β functions get close to zero remaining very flat.



The EFT

We mimick it adding a small mass for the dilaton.

Consider a system with $U(1)$ global symmetry in four dimensions.

$$L[\sigma, \chi] = \frac{f_\pi^2 e^{-2f\sigma}}{2} \partial_\mu \chi \, \partial_\mu \chi - e^{-4f\sigma} C^4 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \, \partial_\mu \sigma - \frac{\xi R}{f^2} \right) - \frac{m_\sigma^2}{16 f^2} \left(e^{-4f\sigma} + 4f\sigma - 1 \right)$$

m_σ is the mass of σ (around $\sigma = 0$) that is due to the underlying (walking) dynamics.
It measures the breaking of scale invariance

$$T^\mu{}_\mu = \frac{m_\sigma^2}{f} \sigma .$$

What is the dilaton mass?

In the conformal model at fixed charge the fluctuations of the dilaton around the classical solution are heavy.

Very little to do with m_σ, which is a measure of how much the full theory is non-conformal.

In the large charge approach it will appear in the semiclassical ground state energy.
The semiclassical state resums the quantum effects.

The ground state energy

We just need to solve at fixed values of the charge.

The energy in the cylinder frame has a new, characteristic term

$$r_0 E_{\text{cyl}} = \frac{c_{4/3}}{(4\pi^2)^{1/3}} Q^{4/3} + c_{2/3} Q^{2/3} - \frac{\pi^2 m_\sigma^2 r_0^4}{3f^2} \log(Q) + \dots$$

This is the first time that a $\log(Q)$ term appears in this game.

The two-point function

Close to the fixed point, we can still use the state-operator correspondence.

The two-point function on $\mathbb{R}^4$ for operators of fixed charge is

$$\langle \mathcal{O}_Q(0)\mathcal{O}_{-Q}(x)\rangle = \frac{1}{|x|^{2\Delta}}$$

where Δ has a $\log(Q)$ correction with respect to the dimension at the fixed point Δ^*

$$\Delta = \Delta^* \left(1 - \frac{m_\sigma^2}{24 c_{4/3} f^2 \mu^4} \log(Q)\right)$$

This is a clear signature of a light dilaton in the walking dynamics.

Fluctuations

We can also study the fluctuations on top of the semiclassical fixed-charge state.
We find again two modes.

- A massless mode, which is not anymore exactly conformal

$$\omega = \frac{1}{\sqrt{3}}\left(1 + \frac{m_\sigma^2}{9c_{4/3}f^2\mu^4}\right)p$$

- A massive mode which has essentially the same mass as in the CFT case

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

This is the mass of the fluctuation of σ around the VEV.

Summing it up

- The large-charge approach can be used for walking theories.
- We predict a precise signature of a light dilaton in the two-point functions.
- We have shown the mechanism for the simplest theory.
- The construction can be easily generalized to more realistic situations (around the conformal window).