The unreasonable effectiveness of the large charge expansion

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INFN | Torino

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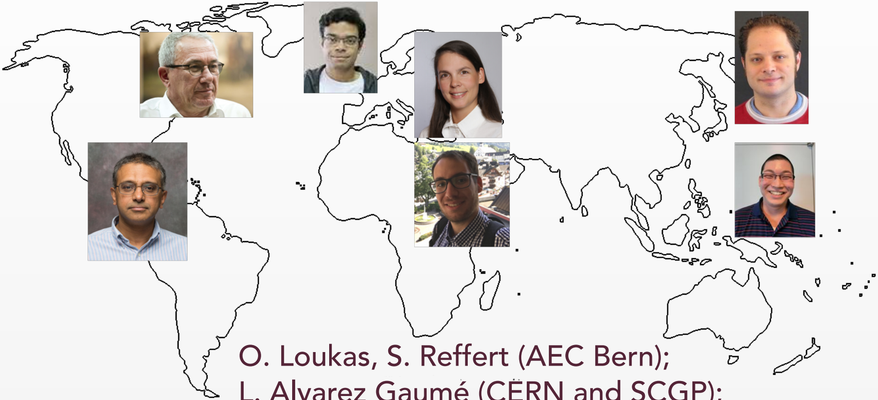
based on [arXiv:1505.01537], [arXiv:1610.04495], [arXiv:1707.00710], [arXiv:1707.00711] and more to come…

Who’s who



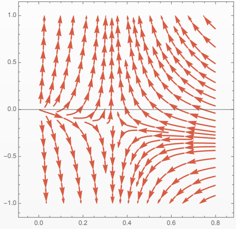
O. Loukas, S. Reffert (AEC Bern);
L. Alvarez Gaumé (CERN and SCGP);
D. Banerjee (DESY);
S. Chandrasekharan (Duke);
S. Hellerman, M. Watanabe (IPMU).

Outline

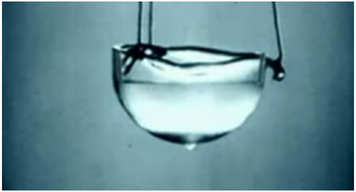
Outline

Why are we here? Conformal field theories

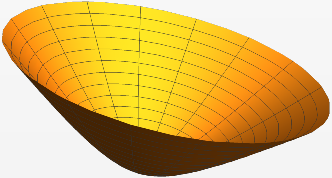
extrema of the RG flow



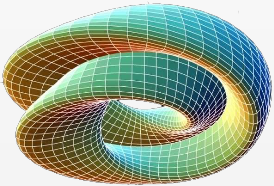
critical phenomena



quantum gravity



string theory

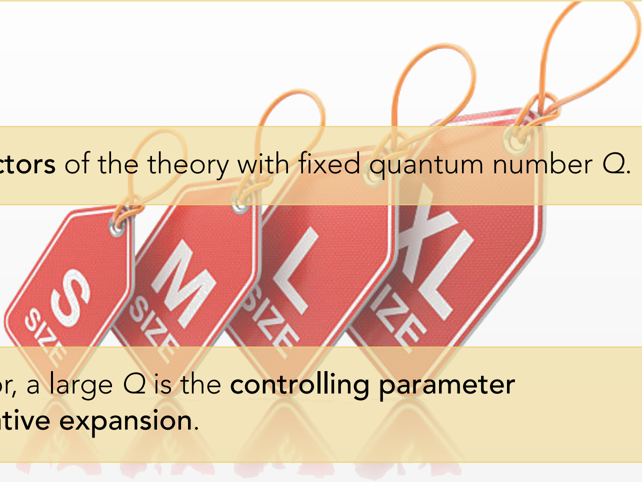


Why are we here? Conformal field theories are hard

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.
No parameter of the theory can be dialed to a simplifying limit.

In presence of a symmetry there can be sectors of the theory where anomalous dimension and OPE coefficients simplify.

The idea



Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.

no bootstrap here!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
But see [arXiv:1710.11161]

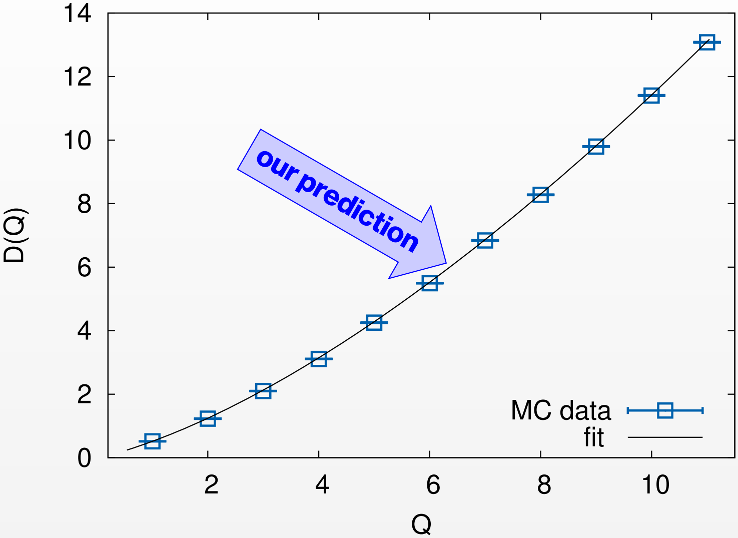
Concrete results

We consider the $O(N)$ vector model in three dimensions. In the IR it flows to a conformal fixed point [Wilson & Fisher].

We find an explicit formula for the dimension of the lowest primary at fixed charge:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

Summary of the results: O(2)



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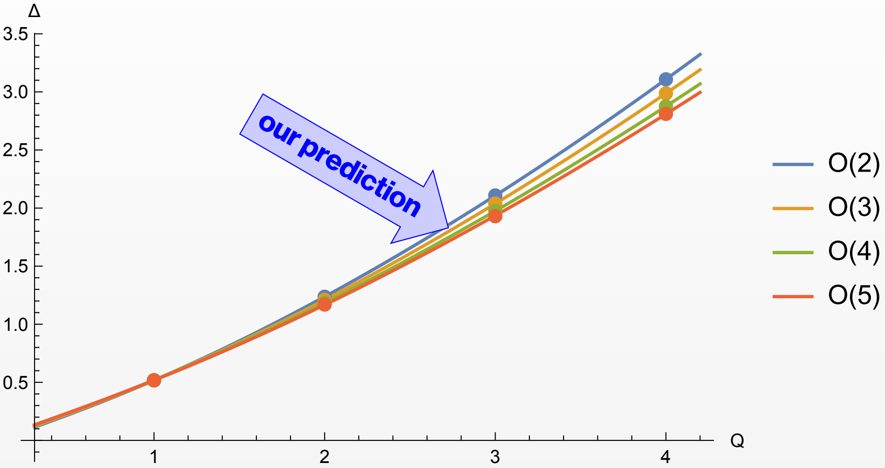

[Hasenbusch and Vicari]

Scales

We want to write a Wilsonian effective action.

Choose a cutoff Λ, separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \, e^{iS(\phi_H, \phi_L)}$$

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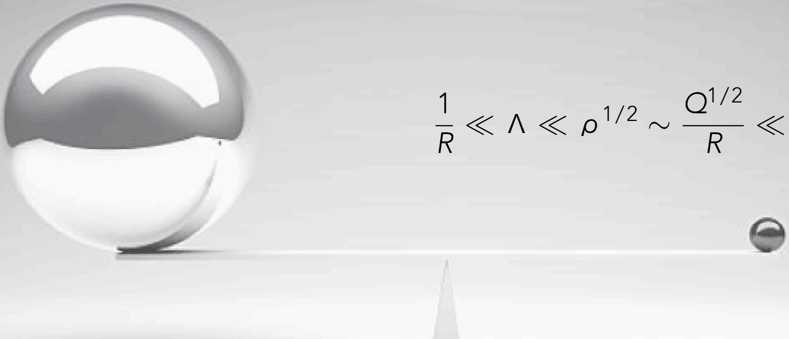
too hard

Scales

- We look at a finite box of typical length R
- The $U(1)$ charge Q fixes a second scale $\rho^{1/2} \sim Q^{1/2}/R$



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

Outline

Introduction

Effective action from classical scale invariance

Quantum analysis

The $O(N)$ model

The UV Lagrangian of the $O(N)$ vector model is of the form

$$\mathscr{L}_{\text{UV}} = \partial_\mu \phi^a \partial^\mu \phi^a - g^2 (\phi^a \phi^a)^2,$$

Wilson and Fisher showed that this flows to a conformal IR fixed point.

$$\text{UV theory} \xrightarrow{\text{RG flow}} \text{IR conformal fixed point.}$$

The idea is to make use of this fact to write an effective Wilsonian action for this universality class.

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forget this

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superstition

Approximate scale invariance

Consider the $O(2)$ universality class.
The order parameter is a complex number $\varphi = a\, e^{ib\chi}$.
Give a large vev to a:

$$\Lambda \ll a^2 \ll g^2.$$

In this limit the Lagrangian is (approximately) scale-invariant with corrections $\sim \Lambda / a^2$.

The IR effective Wilsonian action must be

$$\mathscr{L}_{\text{IR}} = \frac{1}{2}(\partial_\mu a)^2 + \frac{b^2}{2} a^2 (\partial_\mu \chi)^2 - \frac{R}{16} a^2 - \frac{\lambda}{6} a^6 + (\text{higher derivative terms}).$$

where R is the scalar curvature, and b and λ are numerical constants.

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Approximate scale invariance

The charge density is simply

$$\rho := \frac{\delta \mathscr{L}_{\mathrm{IR}}}{\delta \dot{\chi}} = b^2 a^2 \dot{\chi}$$

and using the equations of motion (eom) $a^4 \sim b^2 / \lambda \dot{\chi}^2$ we find that the total on shell charge is

$$Q \sim 4\pi R^2 b \sqrt{\lambda} a^4$$

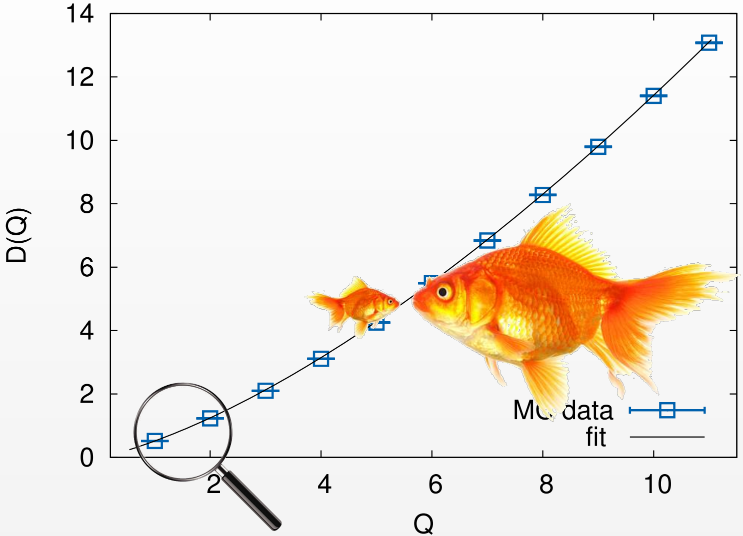
so that the condition $\Lambda \ll a^2 \ll g^2$ on the scales becomes (as promised)

$$\frac{1}{R} \ll \Lambda \ll \frac{Q^{1/2}}{R} \ll g^2$$

which is consistent if the charge is large

$$Q \gg 1.$$

Too good to be true?

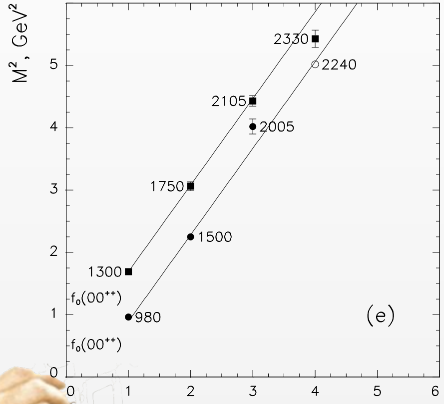


Too good to be true?

Think of Regge trajectories.
The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but *experimentally* everything works so well at small J that String Theory was invented.



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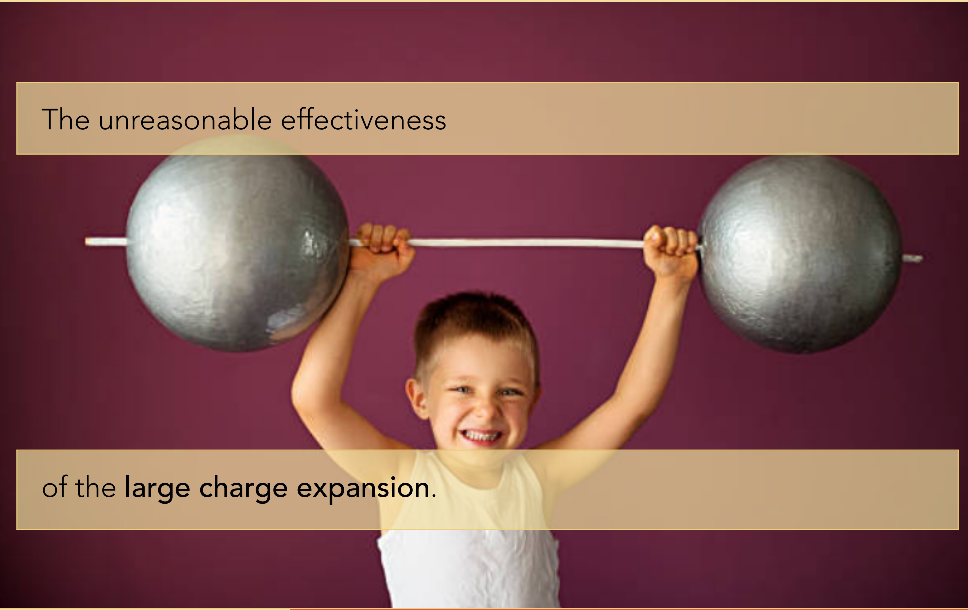

The unreasonable effectiveness

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Outline

Introduction

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Quantum analysis

RG analysis

Now I have to justify my claims:

- Show that the classical solution is precisely of the kind found in the previous slide.
- See how the fluctuations on top of the classical solutions are described by Goldstone modes.
- Show that the higher order terms are suppressed in $1/Q$ for any value of the couplings b and λ.
- Derive the formula for the conformal dimensions.

PARENTAL ADVISORY

EXPLICIT CONTENT

Outline

Outline

Classical analysis

Goldstones

Canonical quantization

Conformal dimensions

Matrix models

Dualities

Conclusions

Abelian global symmetry at fixed charge

Consider a classical system described by Hamiltonian H with a conserved Abelian global symmetry:

$$\{H, Q\} = 0 .$$

we impose the first-class constraint

$$Q = \int \rho \, dx = \overline{Q} = \text{const} .$$

and the corresponding gauge transformation $\delta_\varepsilon f = \{f, \varepsilon Q\}$.
Introduce the canonical conjugate χ to the density ρ

$$\{\chi, Q\} = 1 , \qquad \text{so that } \delta_\varepsilon \chi = \varepsilon ,$$

and assume all the other variables (p_i, q_i) to be gauge invariant.

Abelian global symmetry at fixed charge

For concreteness, consider a natural Hamiltonian system:

$$H = \frac{1}{2}\sum_{k=0}^{N} f_k(q) p_k^2 + \frac{1}{2}\sum_{k=0}^{N} g_k(q)(\nabla q_k)^2 + V(q).$$

We want to find the ground state of this system.
The Hamiltonian is a sum of positive terms, we minimize them separately.
Because of the constraint, $\rho \neq 0$, but we are free to set

$$\nabla q_i = 0, \quad \nabla \chi = 0, \quad p_i = 0, \quad i = 1, \dots, N.$$

Since nothing depends on the position anymore, the constraint becomes

$$\int \rho \, dx = \text{vol.} \times \bar{\rho} = \overline{Q}.$$

Abelian global symmetry at fixed charge

The remaining eom are

$$\dot{p}_i = 0\,,$$
$$\dot{q}_i = 0\,,$$
$$\dot{\chi} = f_0(q_i)\,\bar{\rho}\,.$$

They are solved by

$$p_i = 0\,, \qquad q_i = \bar{q}_i(\bar{\rho})\,, \qquad \chi = \mu(\bar{\rho})t\,,$$

where $\bar{q}_i$ and $\mu(\bar{\rho})$ are constants.

This is the generalization of the classical solution we found in the introduction,

$$a^4 \propto \bar{\rho} \qquad \dot{\chi} \propto \bar{\rho}^{1/2}$$

Outline

Variational description

We want to find a state v that minimizes

$$\langle v|H|v\rangle$$

under the constraints

$$\langle v|v\rangle = 1 \quad \text{and} \quad \langle v|\,\rho\,|v\rangle = \bar{\rho}\,.$$

We introduce the Lagrange multipliers E, m and minimize

$$\langle v|H - E_0 - m\,\rho\,|v\rangle\,.$$

The solution is

$$\left(H - E_0 - m\,\rho\,\right)|v\rangle = 0\,.$$

Variational description

To reproduce the classical solution

$$\langle v| \dot{\chi} |v\rangle = \mu ,$$

where μ is the value found earlier. Now

$$\langle v| \dot{\chi} |v\rangle = \langle v|[\chi , H]|v\rangle = m \langle v|[\chi , \rho]|v\rangle ,$$

and since χ, ρ are canonically conjugate, we obtain

$$m = \mu .$$

The quantum Hamiltonian is given by

$$\mathcal{H} = H - \mu \rho - E_0 .$$

μ is now a fixed chemical potential. The vacuum satisfies $\mathcal{H} |v\rangle = 0$.

Goldstones

The chemical potential breaks explicitly
the symmetry of H from G to $G' \subset G$

$$\mathcal{H} = H - \mu\, \rho\,.$$

The ground state $|v\rangle$ breaks spontaneously to $G'' \subset G'$.
Goldstone tells us: $\dim(G'/G'')$ low energy massless DOF.

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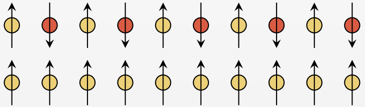
We have singled out the time. The system is non-relativistic.



antiferromagnet $\omega \propto p$

ferromagnet $\omega \propto p^2$ (count double)

A classical vector $O(2n)$ model

Consider the Lagrangian of a $O(2n)$ vector model on $\mathbb{R} \times \Sigma$

$$\mathscr{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} V(\phi_a \phi_a), \qquad a = 1, \dots, 2n,$$

We introduce complex variables

$$\varphi_1 = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2), \dots$$

so the $O(2)^n \subset O(2n)$ generators act as rotations:

$$\{\varphi_i, \varepsilon_j Q_j\} = \varepsilon_j \delta_{ij} \varphi_i \qquad \text{(no sum)}.$$

We impose the conditions

$$\int_\Sigma \mathrm{dvol}\, \rho_i = \overline{Q}_i = V \times \bar{\rho}_i,$$

where the $\bar{\rho}_i$ are fixed.

Ground state

Surprise! The homogeneous ground state solution is

$$\varphi_i = A_i e^{i\mu t}$$

where A_i and μ depend on the fixed charges $\bar{\rho}_i$.
The phase μ is the same for all fields, even if all the charges $\bar{\rho}_i$ are different.

We are really fixing only one $O(2)$ charge – the values of ρ tell us how this is embedded in the maximal $O(2)^n$ torus.

We might as well rotate the solution to

$$\begin{cases} \varphi_i = A_i & i = 1, \dots, n-1 \\ \varphi_n = A_i e^{i\mu t} \end{cases}$$

The classical solution

In the IR the theory becomes conformal (Wilson–Fisher).
The Lagrangian is approximately scale invariant and the potential must have the form

$$V(\| \phi \|) = \frac{R}{16} \| \phi \|^2 + \frac{\lambda}{3} \| \phi \|^6,$$

The classical ground state at fixed charge has energy

$$E_\Sigma(Q) = \frac{c_{3/2}}{\sqrt{V}} Q^{3/2} + \frac{c_{1/2}}{2} R\sqrt{V} Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right),$$

- there are two universal parameters: $c_{3/2}$ and $c_{1/2}$ (*viz.* b and λ)
- the result depends on the manifold Σ only via the volume V and the scalar curvature R

How do the higher derivatives and quantum corrections change this result? How controlled is our approximation?

How many Goldstones?

Using the variational approach, the quantum Hamiltonian is

$$\mathcal{H} = H - \mu(\rho_1 + \rho_2 + \cdots + \rho_k),$$

This breaks the $O(2n)$ symmetry explicitly to $U(n)$. The vacuum

$$\langle \varphi_i \rangle = A_i,$$

breaks $U(n)$ spontaneously to $U(n-1)$.
The dimension of the coset is

$$\dim G/H = \dim U(n) - \dim U(n-1) = 2n - 1.$$

The system is non-relativistic. This is only an upper bound on the number of Goldstones.

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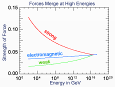

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Expand around the classical solution.

$$\begin{cases} \varphi_i = e^{i\mu t}\hat{\varphi}_i, & i = 1, \dots, n-1 \\ \varphi_n = \frac{1}{\sqrt{2}} e^{i\mu t + i\hat{\phi}_{2n}/v}\left(v + \hat{\phi}_{2n-1}\right) \end{cases}$$

The (unbroken) $U(n-1)$ symmetry is then realized as $\hat{\varphi}_i \mapsto \tilde{U}_i^j \hat{\varphi}_j$.
The second order Lagrangian becomes:

$$\mathcal{L}^{(2)} = \sum_{i=1}^{n} (\partial_t - i\mu)\varphi_i^* (\partial_t + i\mu)\varphi_i - \sum_{i=1}^{n} \nabla\varphi_i^* \nabla\varphi_i - \sum_{i=1}^{n} \mu^2 \varphi_i^* \varphi_i - \frac{2c^2}{1-c^2}\mu^2 \phi_{2n-1}^2,$$

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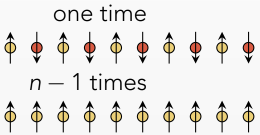
The massless modes are:

$$\omega_r^2 = c^2 p^2 + \frac{\left(1 - c^2\right)^3 p^4}{4\mu^2} + \mathcal{O}\left(\mu^{-4}\right)$$

$$\omega_{nr}^2 = \frac{p^4}{4\mu^2} - \frac{p^6}{8\mu^4} + \mathcal{O}\left(\mu^{-6}\right)$$



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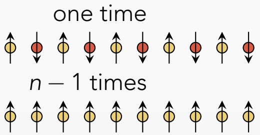

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Non-relativistic ones "count double" [Nielsen and Chadha] [Murayama and Watanabe] and we have $2 \times (n-1) + 1 = 2n - 1 = \dim G/H$.

Outline

Canonical quantization of the non-Abelian sector

The quadratic Hamiltonian in the  sector

$$H_i = \pi_i^* \pi_i + \nabla \varphi_i^* \nabla \varphi_i + \mu^2 \varphi_i^* \varphi_i - \mu (\pi_i \varphi_i - \pi_i^* \varphi_i^*) .$$

Go to Fourier space and expand in terms of canonical operators:

$$\varphi_i(p) = \frac{1}{\sqrt{2\tilde{\omega}(p)}} (a_i(p) + b_i^\dagger(-p)) ,$$

The Hamiltonian is diagonalized by the choice $\tilde{\omega}^2 = p^2 + \mu^2$:

$$H_i(p) = \left(\sqrt{p^2 + \mu^2} - \mu \right) a_i^\dagger(p) a_i(p) + \left(\sqrt{p^2 + \mu^2} + \mu \right) b_i^\dagger(p) b_i(p) .$$

We have broken Lorentz invariance, and the symmetry between particles and antiparticles.

For $\mu \gg 1$, a is a Goldstone with $\omega \sim \frac{p^2}{2\mu}$ and b is massive.

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Non-relativistic Goldstones

Write the Lagrangian

$$\mathcal{L}_i = (\partial_t - i\mu)\varphi_i^*(\partial_t + i\mu)\varphi_i - \mu^2\varphi_i^*\varphi_i - \nabla\varphi_i^*\nabla\varphi_i .$$

If $\mu \gg \partial_t$, this is a massless Schrödinger particle:

$$\mathcal{L}_i = i\mu(\dot{\varphi}_i^*\varphi_i - \varphi_i^*\dot{\varphi}_i) - \nabla\varphi_i^*\nabla\varphi_i ,$$

The term $\mu(\rho_1 + \cdots + \rho_k)$ is a Berry's phase and we get only one classical Goldstone particle instead of two (ferromagnet).

φ and φ^* are canonically conjugate to each other. The Goldstones "count double".
Non-relativistic Goldstones do not contribute to the Casimir energy.

The Abelian sector

The Hamiltonian for the  sector (where the mass term appears) is

$$H_n = \frac{1}{2}\Big[\pi_{2n-1}^2 + \pi_{2n}^2 + (\nabla\phi_{2n-1})^2 + (\nabla\phi_{2n})^2 \\ + \mu^2\left(\frac{1+3c^2}{1-c^2}\phi_{2n-1}^2 + \phi_{2n}^2\right) - \mu\left(\pi_{2n-1}\phi_{2n} - \pi_{2n}\phi_{2n-1}\right)\Big].$$

Also this can be diagonalized in the oscillators:

$$H_n = c\,p\,a_n^\dagger(p)a_n(p) + \frac{2\mu}{\sqrt{1-c^2}}\,b_n^\dagger(p)b_n(p) + \mathcal{O}\left(\frac{1}{\mu}\right).$$

We see that a is a Goldstone with $\omega = cp$ and b is massive.

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Suppression of the interactions

We have assumed that the quadratic part of the Hamiltonian is the most important and that the rest can be treated as small.
Expand the potential:

$$V(\phi) = V(v^2) + \mu^2 \lambda^{i_1 i_2} \varphi_{i_1} \varphi_{i_2} + \mu^2 \frac{\lambda^{i_1 i_2 i_3}}{v} \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} + \dots \\ + \mu^2 \frac{\lambda^{i_1 \dots i_m}}{v^{m-2}} \varphi_{i_1} \dots \varphi_{i_m},$$

where the λ are dimensionless constants and of order $\mathcal{O}(1)$.
To diagonalize H_2, φ_i is of order $\mathcal{O}(\mu^{-1/2})$ so

$$\frac{\mu^2 \lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2}} = \frac{\lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2-2}}.$$

v has the dimensions of a field, $[v] = d/2 - 1$. Overall we have

$$\frac{\lambda^{i_1 \dots i_m}}{\mu^{-d+m/2(d-1)}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{(m/2-d/(d-1))}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{\Omega_m}} \qquad \Omega_m > 0.$$

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The leading effective action

Upshot? The large-charge dynamics is controlled by a single relativistic Goldstone.

- We can integrate out all the massive modes
- The non-relativistic Goldstones are controlled by negative powers of Q

We can write an effective action for the controlling mode

$$S = \int d^3x \sqrt{g}\left[k_{3/2}\|\partial\chi\|^3 + k_{1/2}R\|\partial\chi\|\right] + \mathcal{O}\left(Q^{-1/2}\right)$$

where $\|v\| = (v_\mu v^\mu)^{1/2}$.
This action has to be expanded around $\chi = \mu t$.

The point

Let me stop for a moment.

We found:

- A classical solution that in a large-Q expansion starts with $Q^{3/2}$ and contains only terms with semi-integer powers (no Q^0)
- All the interaction terms have negative Q-scaling
- The only term that has Q^0 scaling is the kinetic term for the Goldstones
- There is only one relativistic Goldstone field 

The point

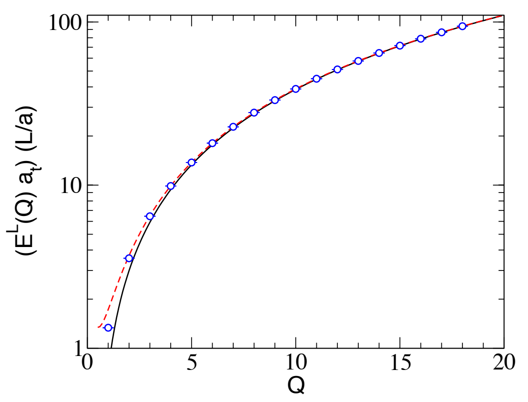
- All the positive Q-scaling terms come from the classical solution
- The only Q^0 term comes from the Casimir energy of the relativistic Goldstone. We only need to compute the ζ function on the manifold of choice.
- Everything else is suppressed for large Q.

Does this work? A small (big) surprise

On a torus $\Sigma = T^2$, the prediction is that the energies go like

$$E_{T^2} = \frac{c_{3/2}}{L} Q^{3/2} + c_0^{T^2} + \mathcal{O}\left(Q^{-1}\right)$$

c_0 is the Casimir energy of our relativistic Goldstone $c_0 = -0.504/L$



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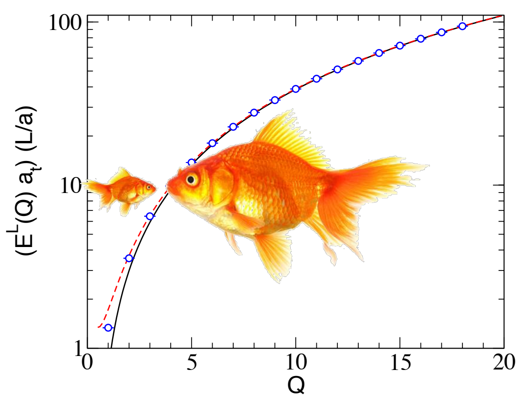

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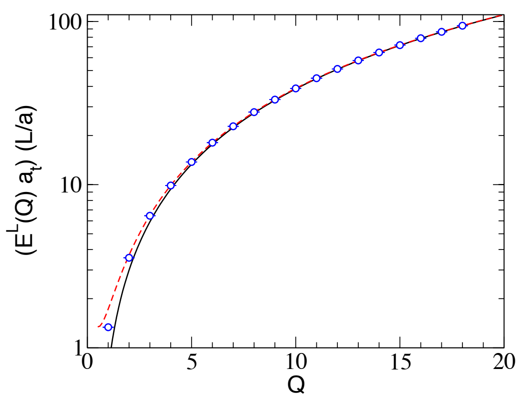

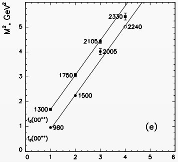

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The point

- We started with a generic $O(2n)$-invariant model
- Fixing n $U(1)$ charges breaks the symmetry explicitly to $U(n)$. We have a controlling parameter $\bar{\rho}$.
- The ground state breaks spontaneously to $U(n-1)$
- There is one relativistic Goldstone (with $c < 1$) and $n-1$ non-relativistic Goldstones, controlled by $\bar{\rho}^{-1}$.
- We diagonalize the quantum Hamiltonian
- In the resulting theory, couplings λ in the initial model are suppressed by powers of $\bar{\rho}^{-1}$.
- In the limit of $\bar{\rho} \to \infty$, the system is well described by a single Goldstone mode.

Outline

Radial quantization

I have promised to compute the conformal dimensions. Up to this point I have computed energies. How are these related?

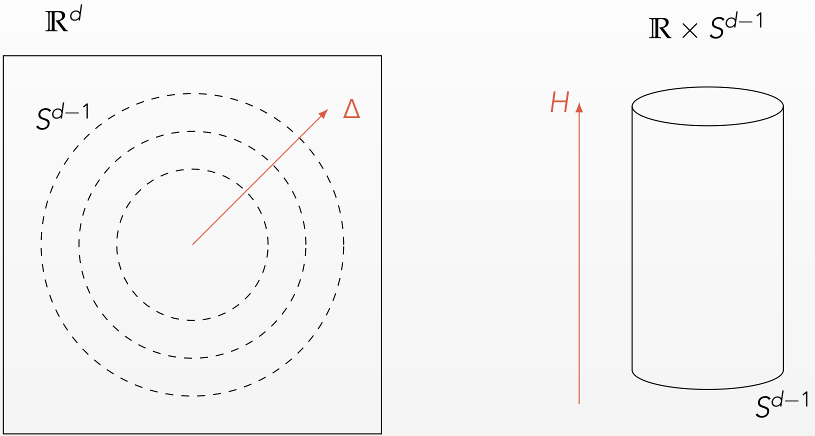
We want to describe a conformal theory, so we can start from flat space $\mathbb{R}^d$and perform a conformal transformation to $\mathbb{R} \times S^{d-1}$:

$$\mathrm{d}s^2 = \mathrm{d}\,\tau^2 + \mathrm{d}\Omega_{d-1}^2 = \frac{1}{r^2}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-1}^2\right),$$

The initial time coordinate has now become the radius r and the Hamiltonian is identified with the dilatation operator.
A state with fixed charge and energy E on $\mathbb{R}_t \times S^{d-1}$ is mapped to an operator on $\mathbb{R}^d$ with conformal dimension

$$\Delta = E.$$

Radial quantization



The action

Up to higher-derivative terms the action must be:

$$S = \frac{1}{2} \int \mathrm{d}t \, \mathrm{d}\Omega \left[g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a - V(\phi^a \phi^a) \right],$$

where the potential becomes now

$$V(\phi^a \phi^a) = \sum_{a=1}^{2n} \left(\frac{R}{8} (\phi^a)^2 + \frac{\lambda}{3} (\phi^a)^6 \right),$$

R is the Ricci scalar $R = 2$.
Naturalness implies $\lambda = \mathcal{O}(1)$, so no standard perturbation theory.

In the limit of large charge, we have a single Goldstone mode and the quantum corrections are controlled by $\lambda / \overline{Q}^{\#} \ll 1$.

Energies

We need just to evaluate the energy of the ground state:

$$E_0 = 4\pi \left(\frac{2\lambda^{1/4}}{3b^{3/2}} \bar{\rho}^{3/2} + \frac{R}{16 b^{1/2} \lambda^{1/4}} \sqrt{\bar{\rho}} + \mathcal{O}\left(\bar{\rho}^{-1/2}\right) \right).$$

The effect of the Goldstone is of order $\mathcal{O}(Q^0)$ and is the one-loop vacuum energy. One just needs to compute a determinant:

$$\log\det\left(-\partial_0^2 + \frac{1}{2}\nabla^2\right) = \frac{1}{\sqrt{2}} \sum_{l=0}^{\infty} (2l+1)\sqrt{l(l+1)}$$

which is ζ-function regularized:

$$E_G \simeq \frac{1}{2\sqrt{2}} \left(-\frac{1}{4} - 0.015\right) = -0.094.$$

This is a universal prediction for our construction.

Conformal dimensions

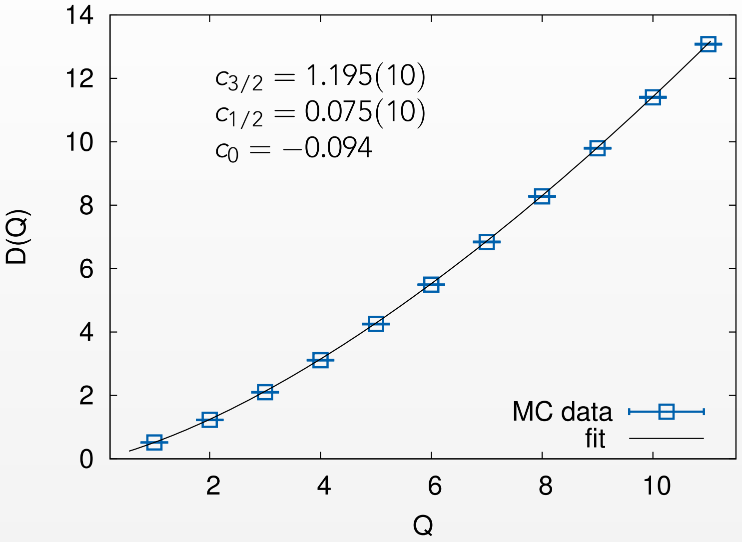
We can put it all together

$$\begin{aligned}\Delta_{Q} &= E_0 + E_G \\ &= \frac{c_{3/2}}{2\sqrt{\pi}} \overline{Q}^{3/2} + 2c_{1/2}\sqrt{\pi}\,\overline{Q}^{1/2} - 0.094 + \mathcal{O}\left(\overline{Q}^{-1/2}\right).\end{aligned}$$

This is a prediction for the conformal dimensions at the Wilson–Fisher point of the $O(n)$ model.

There are two parameters $c_{3/2}$ and $c_{1/2}$ that depend on the details of the model.
They can be computed *e.g.* on the lattice.

Large charge and the lattice



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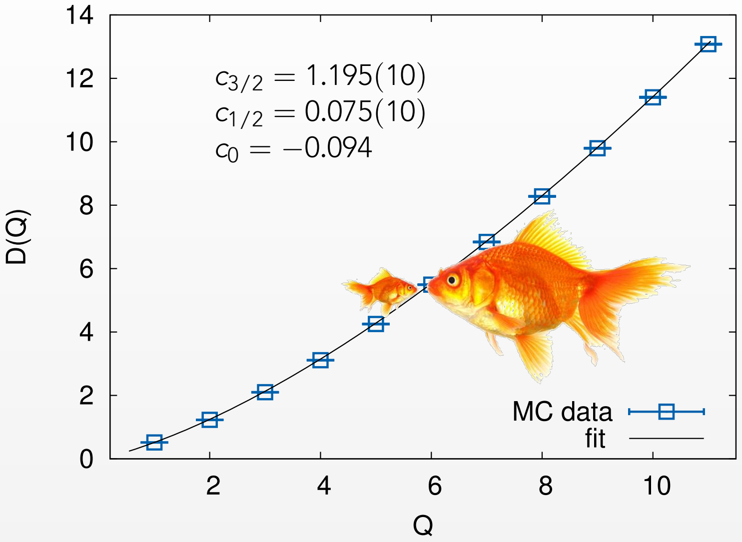

Outline

Matrix models

We have chosen a global symmetry and a (vector) representation. What happens when the fields transform *i.e.* in the adjoint? Is the physics different?
We can start with Let's start with

$$S = \int_{\mathbb{R}\times\Sigma} \mathrm{d}t\, \mathrm{d}\Sigma \left[\frac{1}{2}\operatorname{Tr}(\partial_\mu \Phi\, \partial^\mu \Phi) - V(\Phi)\right],$$

where $\Phi \in A_{N-1}$ and

$$V(\Phi) = \frac{R}{16}\operatorname{Tr}\Phi^2 + g_1 \operatorname{Tr}\Phi^6 + g_2(\operatorname{Tr}\Phi^3)^2 + g_3 \operatorname{Tr}\Phi^4 \operatorname{Tr}\Phi^2 + g_4(\operatorname{Tr}\Phi^2)^3,$$

Matrix models

The analysis is quite similar in spirit to the one above but the large-charge behavior is different [Loukas 1711.07990]

- For $N = 2$ and $N = 3$ there is just one Goldstone, like in the $O(N)$ model
- For $N \geq 4$ the generic potential leads to a different symmetry-breaking pattern where homogeneous solutions can have $\lfloor N/2 \rfloor$ independent charges.

Matrix models

A $U(N)$ vector model and a $U(N)$ matrix model have the same global symmetries and are not a priori different in a bootstrap approach.

Our Wilsonian construction tells us that for $N \geq 4$ they are different even before we study their dynamics.
One admits homogeneous solutions with multiple fixed charges, the other does not.

The large-charge approximation can help in mapping the phase space.

Outline

Large charge and dualities

A favorite approach to strongly coupled theories consists in using dualities. The idea is to find a weakly-coupled dual description of a strongly-coupled theory.

The large-quantum-number limit “classicalizes” strongly coupled theories.

We can hope that in this limit, strong/weak dualities become classical/classical dualities.

The dual of the $O(2)$ model

The simplest situation where such a duality can be described concretely is again the $O(2)$ model in three dimensions.
It is known that there exists a dual description in terms of an Abelian gauge theory.

- The Noether current maps to the monopole current
- The total Noether charge becomes the magnetic flux on the sphere
- The Noether charge of an operator becomes the monopole number

An equivalent action

We have an effective action for the $O(2)$ model at fixed charge

$$S = \int d^3x \sqrt{g} \|\partial \chi\|^3 .$$

We want to dualize it to the theory of a two-form.
The trick is to rewrite the action using a field $v_\mu = \partial_\mu \chi$ and impose that $dv = 0$ with a Lagrange multiplier:

$$S = \int d^3x \sqrt{g} \|v\|^3 + \frac{1}{2\pi} \varepsilon_{\mu\nu\rho} a_\mu \partial_\nu v_\rho$$

The Abelian theory

$$S = \int \mathrm{d}^3x \sqrt{g}\|v\|^3 + \frac{1}{2\pi} \varepsilon_{\mu\nu\rho} a_\mu \partial_\nu v_\rho$$

Integrating out a gives the starting Lagrangian.

Integrating out v gives the Lagrangian for the dual field $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$:

$$S = \int \mathrm{d}^3x \sqrt{g}\|f\|^{3/2}$$

An immediate consequence of the form of the action is that the dimension of the lowest-lying monopole operator scales as the monopole number to the $\frac{3}{2}$ (for large monopole charge).

Monopoles in three dimensions

Identifying the $U(1)$ current on both sides

$$f_{\mu\nu} \propto \|\partial\chi\| \sqrt{|g|}\, \varepsilon_{\mu\nu\sigma} \partial^{\sigma} \chi ,$$

In fact Weyl-invariance, diffeomorphism covariance, and charge quantization uniquely determine this relation.

Fixed charge in the $O(2)$ model becomes a background vacuum expectation value (vev) for the magnetic flux

$$\langle f_{\theta\phi} \rangle = \frac{Q}{2R} \sin\theta$$

(Large) Spin

Adding vortices to the Abelian action is equivalent to adding spin to the operators in the $O(2)$ model [Cuomo et al. 1711.02108].

In the large-charge approximation we can keep only the classical terms (no quantum corrections).

regime	Δ	state
$0 < J < Q^{1/2}$	$\alpha\, Q^{3/2} + \frac{J}{\sqrt{2}}$	no vortices
$Q^{1/2} < J < Q$	$\alpha\, Q^{3/2} + \frac{\sqrt{Q}}{3\alpha} \log \frac{J}{\sqrt{Q}}$	vortex-antivortex pair
$Q < J < Q^{3/2}$	$\alpha\, Q^{3/2} + \frac{1}{2\alpha} \frac{J^2}{Q^{3/2}}$	many vortex-antivortex pairs

Outline

Summary of the results

Very concrete examples where a strongly-coupled CFT is simplified in a special sector.

$O(N)$ model in three dimensions: in the limit of large $U(1)$ charge Q, we computed the conformal dimensions in a controlled perturbative expansion.

We have found an explicit formula for the dimension of the lowest-energy state:

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.094$$

Now what?

- We would like to get a better understanding of the Wilsonian action. In particular we would like to compute the coefficients $c_{3/2}$ and $c_{1/2}$ from first principles;
- Why does the approach work numerically for small charge?

We have described one example.

We hope our framework is powerful enough to provide insights in the large–Q behavior of other strongly coupled CFTs which are in general not tractable with known methods.

Thank you for your attention