Deformations, defects and a noncommutative spectral curve

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Outline

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Introduction and motivation

The fluxtrap

The Ω deformation

Noncommutativity from geometry

Wilson lines and surfaces

The Ω deformation

The Ω background was introduced by Nekrasov as a way of regularizing the four-dimensional instanton partition function and reproducing the results of Seiberg and Witten.

One introduces an appropriate deformation of the four-dimensional theory, with parameters ϵ_1 and ϵ_2, breaking rotational invariance of $\mathbb{R}^4$.

The path integrals localize on a discrete set of points.
The k-instanton contribution to the prepotential for the original (undeformed) theory is found in the limit $\epsilon_i \to 0$.

Philosophy

If a problem is hard, make it harder
(and new structures will appear).

Finite ε

In fact this turned out to be a much richer subject.
The partition function in the Ω background has a meaning also for finite values of ε.

- In the limit $\varepsilon_1 = -\varepsilon_2 \propto g_s$ the partition function is the same as the one for topological strings on a CY related to the spectral curve;
- In the limit $\varepsilon_1 = 0$ the gauge theory is closely related to quantum integrable models with $\hbar = \varepsilon_2$;
- In the general case $\varepsilon_1 \neq \varepsilon_2$, we have the refinement of topological strings;
- The AGT construction can be understood in terms of compactifications of a six-dimensional theory on the Ω background.

The fluxtrap

But there is more.

The string theory background that realizes this deformation has many interesting properties:

- it's an exact CFT
- it's directly related to noncommutativity
- can be used to realize explicitly field theories in presence of defects of different dimensions
- it's the common origin of different gauge theories that seem unrelated.

Outline

Melvin construction in field theory

We want to write the String Theoretical analog to a compactification with a Wilson line. In the Melvin construction one starts with an S^1 fibration over $\mathbb{R}^4$, with a non-trivial monodromy

$$\begin{array}{ccc} S^1(\tilde{u}) & \longrightarrow & M \\ & & \downarrow \\ & & \mathbb{R}^4(\rho_k, \theta_k) \end{array} \qquad \begin{cases} \tilde{u} \sim \tilde{u} + 2\pi n_u , \\ \theta_k \sim \theta_k + 2\pi \varepsilon_k \tilde{R} n_u , \end{cases} \qquad n_u \in \mathbb{Z}$$

T-duality is the string theory version of a reduction on S^1.

The String Theory version

Start from a Ricci-flat metric $ds^2 = g_{ij}\,dx^i\,dx^j + d(\tilde{x}^9)^2$, where $\tilde{x}^9 = \widetilde{R}\tilde{u}$, where g has $N \le 4$ (non-bounded) rotational isometries generated by ∂_{θ_k}.
Pass to a set of disentangled variables

$$\phi_k = \theta_k - \epsilon_k \widetilde{R}\tilde{u}\,,$$

This modifies the boundary conditions from

$$(\tilde{u}, \theta_k) \sim (\tilde{u}, \theta_k) + 2\pi n_u \left(1, \epsilon_k \tilde{R}\right) + 2\pi n_k (0, 1)$$

to

$$(\tilde{u}, \phi_k) \sim (\tilde{u}, \phi_k) + 2\pi n_u (1, 0) + 2\pi n_k (0, 1)\,.$$

The price to pay is the appearance of a graviphoton $\epsilon U_i\,dx^i$.

The generic fluxtrap

Now T-dualize in $\tilde{u}$. We get a B-field and a non-trivial dilaton: the fluxtrap

$$
\begin{aligned}
ds^2 &= g_{ij}\, dx^i\, dx^j - \frac{\varepsilon^2 U_i U_j\, dx^i\, dx^j}{1+\varepsilon^2 U_i U^i} + \frac{(dx^9)^2}{1+\varepsilon^2 U_i U^i}, \\
B &= \varepsilon \frac{U_i\, dx^i \wedge dx^9}{1+\varepsilon^2 U_i U^i}, \\
e^{-\Phi} &= \frac{\sqrt{\alpha'}\, e^{-\Phi_0}}{R} \sqrt{1+\varepsilon^2 U_i U^i},
\end{aligned}
$$

We have taken the limit $\tilde{R} \to 0$: in this picture the irrelevant degrees of freedom (rotations around $\tilde{u}$) have been removed (they turn into infinitely heavy winding modes). All the local degrees of freedom are physical.

The generic fluxtrap

$$ds^2 = g_{ij}\, dx^i\, dx^j + \frac{(dx^9)^2 - \varepsilon^2 U_i U_j\, dx^i\, dx^j}{1 + \varepsilon^2 U_i U^i},$$

$$B = \varepsilon \frac{U_i\, dx^i \wedge dx^9}{1 + \varepsilon^2 U_i U^i},$$

$$e^{-\Phi} = \frac{\sqrt{\alpha'}\, e^{-\Phi_0}}{R} \sqrt{1 + \varepsilon^2 U_i U^i},$$

- For $\varepsilon = 0$ this is the initial Ricci-flat background
- U is the generator of the rotational isometries before and after the duality

$$\varepsilon\, U^i \partial_i = \sum_{k=1}^{N} \varepsilon_k \partial_{\phi_k}$$

- branes will be trapped in $U = 0$ by the terms in the denominators
- ε regularizes the rotation, which is always bounded if $\varepsilon \neq 0$

$$\|U\|^2_{\text{trap}} = \frac{U_i U^i}{1 + \varepsilon^2 U_i U^i} < \frac{1}{\varepsilon^2}.$$

- the dilaton has a maximum when $U = 0$.

Fluxtrap around flat space

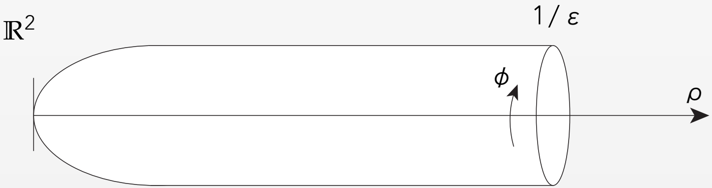
To get an intuitive picture of the deformation, start with flat space and twist in two directions ($\tilde{u}$ and $\tilde{v}$).

$$\begin{cases} \tilde{u} \sim \tilde{u} + 2\pi n_u , \\ \theta_1 \sim \theta_1 + 2\pi \epsilon_1 \tilde{R}_u n_u , \end{cases} \qquad \begin{cases} \tilde{v} \sim \tilde{v} + 2\pi n_v , \\ \theta_2 \sim \theta_2 + 2\pi \epsilon_2 \tilde{R}_v n_v , \end{cases}$$

After two T-dualities, the space takes the form of a product

$$M_{10} = M_3(\epsilon_1) \times M_3(\epsilon_2) \times \mathbb{R}^4$$

where $M_3(\epsilon)$ is a $\mathbb{R}$ fibration (the dual direction) over a cigar with asymptotic radius $1/\epsilon$. The NS three-form is the volume of M_3.



Supersymmetry in type IIA

T–duality maps the Killing spinors η_{iib} into local type iia Killing spinors η_{iia} .
Using an appropriate vielbein for the T–dual metric they take the form $\eta_{\text{iia}} = \eta^L_{\text{iia}} + \eta^R_{\text{iia}}$ with

$$\begin{cases} \eta^L_{\text{iia}} = (\mathbb{1} + \Gamma_{11}) \prod_{k=1}^{N} \exp\left[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k}\right] \mathrm{P}^{\text{flux}} \eta_w , \\ \eta^R_{\text{iia}} = (\mathbb{1} - \Gamma_{11}) \Gamma_u \prod_{k=1}^{N} \exp\left[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k}\right] \mathrm{P}^{\text{flux}} \eta_w , \end{cases}$$

where Γ_u is the gamma matrix in the u direction normalized to unity.

Supersymmetry

Depending on η_W, the projector P^{flux} can either break all supersymmetries or preserve some of them. In the latter case, at least $1/2^{N-1}$ of the original ones are preserved.

Examples:

- In flat space η_W is a constant spinor with 32 independent components. Each independent ε breaks 1/2 of the supersymmetry;
- There are special configurations with 12 supercharges
- In the Taub–nut case, the orientation is fixed by the triholomorphic $U(1)$ isometry:
  - The choice $\varepsilon_1 = -\varepsilon_2$ preserves all supersymmetries.
  - The choice $\varepsilon_1 = \varepsilon_2$ breaks all supersymmetries.

The point

- We look at a String Theory realization of the Melvin construction
- T-duality removes the non-physical degrees of freedom
- We find a background where all local degrees of freedom are physical
- We can study this background using String Theory
- Supersymmetry in terms of Killing spinors in the bulk

Outline

The gauge theory

Now that we have found the bulk we can try to reproduce the Ω-deformed four-dimensional gauge theory. The idea is to place D–branes a la Hanany–Witten, so that the gauge theory encodes their fluctuations.

Consider a stack of D4–branes extended in the directions of the shifts.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane	ε_1		ε_2		ε_3		×	×	×	○
NS5	×	×	×	×					×	×
D4	×	×	×	×			×			
ξ	0	1	2	3			4			

The Ω-deformed action

Now we just need to write the DBI action expanded at second order in the fields:

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(\|F\|^2 + \frac{1}{2}\|\mathrm{d}\varphi + 2\mathrm{i}\,\varepsilon\,\iota_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\iota_{\hat{U}}\mathrm{d}(\varphi+\bar{\varphi})\|^2\right),$$

where $\hat{U}$ is the pullback of the vector field U,

$$\varepsilon\,\hat{U} = \varepsilon\, f^* U = \varepsilon\,\hat{U}^i\,\partial_{\xi^i} = \varepsilon_1\left(\xi^0\partial_1 - \xi^1\partial_0\right) + \varepsilon_2\left(\xi^2\partial_3 - \xi^3\partial_2\right).$$

Lagrangian of the Ω–deformation of $\mathcal{N} = 2$ SYM. [Nekrasov-Okounkov]

The advantage is that now we can understand it as coming from string theory and we have an algorithmic way to generalize it.

The interpretation

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(1 + \|F\|^2 + \frac{1}{2}\| \mathrm{d}\varphi + 2\,\mathrm{i}\,\varepsilon\,\imath_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\imath_{\hat{U}}\mathrm{d}(\varphi + \bar{\varphi})\|^2\right),$$

- the terms in ε are odd under charge conjugation $A_\mu \to -A_\mu$. This is because they come from the B field. This is the leading deformation of the background
- the terms in ε^2 come from metric and dilaton. They control classical gauge configurations and hence directly to the instanton moduli space

A single instanton

- A D–instanton is a $D(-1)$ brane. Its action is

$$\mathscr{L}_{\text{inst}} = e^{-\Phi} + \text{fermions} = \sqrt{1+\varepsilon^2 \|U\|^2} + \frac{\psi^\mu \bar{\omega}_{\mu\nu} \psi^\nu}{\sqrt{1+\varepsilon^2\|U\|^2}}$$

- a critical point for the action is a critical point for the dilaton profile: $U = 0$. This is the string theoretical version of localization.
- These are moduli, so the path integral is just a standard integral

$$\begin{aligned} I &= \int d^{2k}x\, d^{2k}\psi \exp[-\mu S] \\ &= \int d^{2k}x \exp\left[-\mu\sqrt{1+U^2}\right] \frac{\mu^k}{2^k(1+U^2)^{k/2}} \prod_{l=1}^{k} \bar{\varepsilon}_l \\ &= \frac{N_k(\mu)}{\prod_{l=1}^k \varepsilon_l} = \frac{4N_4(\mu)}{\varepsilon_1\varepsilon_2(2m-\varepsilon_1-\varepsilon_2)(2m+\varepsilon_1+\varepsilon_2)} \end{aligned}$$

The point

- We realize the deformed four-dimensional gauge theory in terms of Hanany–Witten branes (D4 suspended between NS5)
- The fluxtrap background is pulled back on the branes and modifies the theory
- We have a geometric origin for the new terms in the action
- Localization can be understood in terms of dilaton gradient

Outline

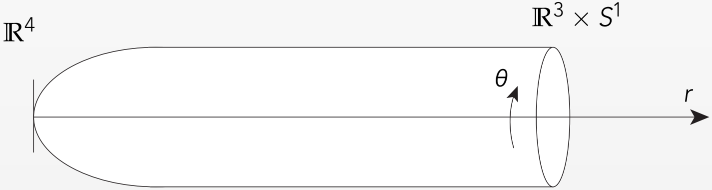
Taub–nut

In order to make our construction more transparent it is convenient to start from a Taub–nut space and put a fluxtrap in $\mathrm{TN}_\Omega \times S^1 \times \mathbb{R}^5$.

$$S^1(\theta) \longrightarrow \mathrm{TN}$$
$$\downarrow$$
$$\mathbb{R}^3(\mathbf{r})$$

A Taub–nut space is a singular S^1 fibration over $\mathbb{R}^3$

It interpolates between $\mathbb{R}^4$ for $r \to 0$ and $\mathbb{R}^3 \times S^1$ for $r \to \infty$.



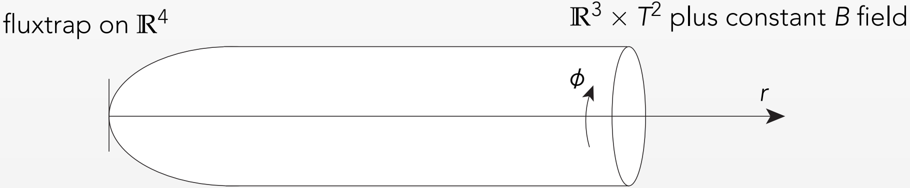
Flux–trap

$$ds^2 = V(r)\,dr^2 + \frac{1}{V(r) + \epsilon^2}\left(d\phi + Q\cos\omega\, d\psi\right)^2 + \frac{V(r)}{V(r)+\epsilon^2}(dx^9)^2 + d\mathbf{x}^2_{4\dots8}\,,$$

$$B = \frac{\epsilon}{V(r)+\epsilon^2}\left(d\phi + Q\cos\omega\, d\psi\right)\wedge dx^9\,,$$

$$e^{-\Phi} = \sqrt{1 + \frac{\epsilon^2}{V(r)}}\,.$$

This interpolates between the fluxtrap in flat space that we used to reproduce Nekrasov's action and $\mathbb{R}^3 \times T^2$ with a constant B field.



The alternative description

In the limit $r \to \infty$ the Taub–nut becomes $\mathbb{R}^3 \times S^1$ and the fluxtrap is the result of a T–duality on a torus with shear, *i.e.* a constant B field.

Putting a D4–brane wrapping the Taub–nut space we obtain the alternative description of the Ω deformation proposed by Witten and Nekrasov.

Noncommutativity

The Ω deformation for $\varepsilon_1 = -\varepsilon_2$ is related to topological strings.
It has been observed that

> *the Riemann surface Σ behaves for many purposes as a subspace of a quantum mechanical (s, v) phase space where $g_s = \hbar$.* [Aganagic, Dijkgraaf, Klemm, Marino, Vafa]

> *this gauge theory provides the quantization of the classical integrable system underlying the moduli space of vacua of the ordinary four dimensional $N = 2$ theory* [Nekrasov, Shatashvili]

Our construction gives a precise geometrical interpretation for this observation in terms of Riemann surface on a non-commutative plane.

Reduction

Lift the background to M-theory... and reduce it on ϕ

$$ds^2 = V(r)^{1/2}\, d\mathbf{r}^2 + V(r)^{-1/2}\left[d\mathbf{x}^2_{4\dots10} - \frac{\varepsilon^2}{V(r)+\varepsilon^2}\left((dx^9)^2 + (dx^{10})^2\right)\right],$$

$$B = \frac{\varepsilon}{V(r)+\varepsilon^2}\, dx^9 \wedge dx^{10},$$

$$e^{-\Phi} = V(r)^{1/4}\sqrt{V(r)+\varepsilon^2},$$

$$C_1 = Q\cos\omega\, d\psi\,,$$

$$C_3 = \frac{\varepsilon\, Q\cos\omega}{V(r)+\varepsilon^2}\, dx^9 \wedge dx^{10} \wedge d\psi\,.$$

These are Q D6–branes extended in $(x^4, \dots, x^{10})$ in presence of an Ω–deformation.

The Seiberg–Witten map

An equivalent description is obtained by applying the Seiberg–Witten map to the D6–brane theory in order to turn the B–field into a non-commutativity parameter:

$$(\hat{g} + \hat{B})^{-1} = \tilde{g}^{-1} + \Theta ,$$

where $\hat{g}$ and $\hat{B}$ are the pullbacks of metric and B–field on the brane and $\tilde{g}$ is the new effective metric for a non-commutative space satisfying

$$\left[x^i, x^j\right] = \mathrm{i}\, \Theta^{ij} .$$

Applying this map to our case:

$$\tilde{g}_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \mathrm{d}\mathbf{x}^2_{4\ldots10} ,$$
$$\left[x^9, x^{10}\right] = \mathrm{i}\, \varepsilon .$$

All dependence on ε disappears from the D6–brane theory and is turned into a constant non-commutativity parameter.

A non-commutative Riemann surface

Let's follow the fate of the branes whose dynamics reproduce the Ω–deformed gauge theory.

Start from the configuration of D4–NS5s, with the D4 wrapping the Taub–nut space.

In the M–theory lift this configuration turns into a single M5–brane extended in the directions $(x^0, \dots, x^3)$ and wrapped on a Riemann surface Σ embedded in the (s, v) plane.

Reduction on ϕ turns the M5–brane into an D4–brane extended in **r** and wrapped on Σ, which is now embedded in the worldvolume of the D6–brane.
For finite ϵ the Riemann surface Σ is embedded in a non-commutative complex plane where

$$[s, v] = \mathrm{i}\ \epsilon\ .$$

The point

- We repeat our construction starting from a Taub–nut space in the bulk
- The Taub–trap solution interpolates between Nekrasov's original description and Nekrasov–Witten's "alternative" description
- We lift the IIA background to M–theory
- We reduce it on the isometry circle.
- The resulting D6 background has a natural non–commutativity ε
- The gauge theory describes the dynamics of a D4 wrapped on a Riemann surface living on a non-commutative $\mathbb{C}^2$ plane. This is the geometric interpretation of the "quantum spectral curve".

Outline

The RR fluxtrap

Let's look a bit closer at the ten-dimensional background after S-duality.

$$\begin{aligned} ds_{10}^2 &= \Delta \left[-(dx^0)^2 + \frac{(dx^1)^2}{\Delta^2} + \left(\delta_{IJ} - \frac{U_I U_J}{\Delta^2} \right) dx^I dx^J \right] , \\ e^{\Phi} &= \Delta , \\ C_2 &= \frac{1}{\Delta^2} dx^1 \wedge U , \end{aligned}$$

where $\Delta^2 = 1 + U_I U^I$. This is a compact form for writing up to 4 ε parameters (U is the generator of the $U(1)$: $dU = \sum_A \varepsilon_A \, dz_A \wedge d\bar{z}_A$).

What happens if, instead of putting the Ω-deformation branes, we consider different probe brane configurations in the same bulk geometry?

The Wilson line

The simplest configuration is the one of a D1-brane:

x	0	1	2	3	4	5	6	7	8	9
fluxtrap		$\circ$	ε_1		ε_2		ε_3		ε_4	
D1–brane	$\times$	$\times$	Z^1		Z^2		Z^3		Z^4	

The DBI action reads:

$$S_{D1} = -\frac{1}{2g^2}\int d^2x \left[\frac{1}{2}F^2 + \sum_{k=1}^{4}\left(\partial_\mu Z^A + i\,\varepsilon_A B_\mu Z^A\right)\left(\partial^\mu \bar{Z}^A - i\,\varepsilon_A B^\mu \bar{Z}^A\right)\right],$$

where $B_\mu = \delta_\mu{}^0$.

This is a gauge theory in presence of a background Wilson line!

Wilson line

What has happened?

$$S_{D1} = -\frac{1}{2g^2}\int d^2x \left[\frac{1}{2}F^2 + \sum_{k=1}^{4}\left(\partial_\mu Z^A + i\,\varepsilon_A B_\mu Z^A\right)\left(\partial^\mu \bar{Z}^A - i\,\varepsilon_A B^\mu \bar{Z}^A\right)\right],$$

- the contribution at first order in ε comes from the Ramond–Ramond (rr) flux in the bulk via the Chern–Simons (cs) term;
- the metric and the dilaton contribute to the quadratic term.

The same flux that we had described as a noncommutativity for D4 branes now is a Wilson line.

Wilson line

In gauge theory language we have gauged the $U(1)$ symmetries that rotate the four complex fields and given them a time-like vacuum expectation value (vev). Technically we have a new covariant derivative

$$D_\mu = \partial_\mu + B_\mu$$

Geometrically, we break the $SO(1,d)$ symmetry to $SO(d)$ and the undeformed theory is coupled to a one-dimensional defect extended in the time direction.

From the string theory we know that the configuration preserves a number of supersymmetries that depends on how many ε are non-zero.

3d defects

Set $\varepsilon_1 = -\varepsilon_2$ and all the others to zero.
Now T-dualize twice and the bulk fields become

$$ds^2 = \Delta\left(\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta + \delta_{IJ}\, dx^I\, dx^J\right) + \frac{\delta_{ab}\, dx^a\, dx^b - U_I U_J\, dx^I\, dx^J}{\Delta},$$

$$C_4 = U \wedge \left(-\, dx^0 \wedge dx^1 \wedge dx^5 + \frac{dx^2 \wedge dx^3 \wedge dx^4}{\Delta^2}\right),$$

The dilaton has disappeared and we obtain a solution that has only metric and 5-form flux.

Remember: this is still just a few dualities away from flat space with identifications, which is an exact string theory solution.
In principle we can have complete worldsheet control beyond supergravity.

3d defects

Now add a probe D5 brane extended in $\{x^0, \dots, x^5\}$.

$$S_B^{D5} = \int d^6x \Big[-\frac{1}{2}(1 - U_I U_J)\partial_\mu X^I \partial^\mu X^J - \frac{1}{2} U_J U^J \partial_a X^I \partial^a X^I - \frac{1}{2} \Xi_{\mu\nu\rho} F^{\nu\rho} \omega_{IJ} X^J \partial^\mu X^I - \frac{1}{2} F_{\alpha a} F^{\alpha a} - \frac{1}{4}(1 - U_I U^I) F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{4}(1 + U_I U^I) F_{ab} F^{ab} + \mathcal{O}\left(\varepsilon^3\right) \Big],$$

where $\mu = 0, \dots, 6$, $\alpha = 0, 1, 2$, $a = 3, 4, 5$.

How do I read this?

A simple geometric way to understand the meaning of this configuration is to look at the Wess–Zumino (wz) term (leading deformation).

Start with the Wilson line configuration. There:

$$S_{wz} = \int \hat{U} \wedge dx^1$$

this is a defect extended in time.

In the D6 brane case, the wz term reads

$$S_{wz} = \int F \wedge \hat{U} \wedge \left(dx^0 \wedge dx^1 \wedge dx^2 - dx^3 \wedge dx^4 \wedge dx^5\right)$$

It is clear that we are breaking $SO(1,5)$ into $SO(1,2) \times SO(3)$.

We have added two orthogonal three-dimensional defects.

Covariant derivative

For Wilson lines we had found a covariant derivative $D_\mu = \partial_\mu + B_\mu$. What about here?

Again look at the wz term. We can rewrite it as

$$\int \mathrm{d}^6 x \, \partial^\mu X^I \mathcal{A}_{\mu\, IJ} X^J$$

where $\mathcal{A}$ is a connection.

$$\mathcal{A}_{\mu\, IJ} = \frac{1}{2} \Xi_{\mu\,\nu\,\rho} F^{\nu\,\rho} \omega_{IJ}$$

and $\Xi = \epsilon_{\alpha\beta\gamma} + \epsilon_{abc}$ is the sum of the volume forms on the two defects.

It's like having a non-minimal coupling. Very convenient for the supersymmetric analysis.

Defects

In gauge theory language we have a new covariant derivative

$$D_{\mu\, IJ} = \partial_\mu\, \delta_{\,IJ} + \mathcal{A}_{\mu\, IJ}$$

Geometrically, we break the $SO(1,5)$ symmetry to $SO(1,2) \times SO(2)$ and the undeformed theory is coupled to a three-dimensional defect.

From the string theory we know that the configuration preserves a number of supersymmetries that depends on how many ε are non-zero.

The point

- We can probe the same string background using different branes
- The same bulk fields that lead to the noncommutativity now correspond to background Wilson lines
- In different frames we obtain extended defects of codimension 2 and 3
- All the configurations are supersymmetric and the amount of supersymmetry can be controlled via the ε parameters.
- Interesting ways of deforming supersymmetric gauge theories.
- Useful to study non-perturbative physics, dualities...

Conclusions

- We started with a string realization of the Ω deformation
- It is deeply related to noncommutativity
- It is (dual to) an exact string theory
- When probed with different branes it describes defects of different dimension
- All the configurations that we have seen are by construction supersymmetric.
- The string theoretical description is particularly simple and helpful in the study of the gauge theory properties.

Thank you for your attention