

Attacking strong coupling with large charge

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based on [arXiv:1505.01537], [arXiv:1610.04495] and more to come...

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Who's who



Outline

Introduction

Effective action from classical scale invariance

Quantum analysis

Outline

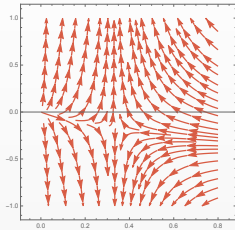
Introduction

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Why are we here? Conformal field theories

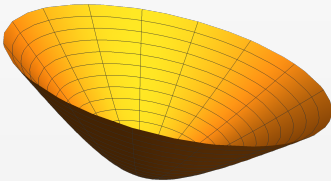
extrema of the RG flow



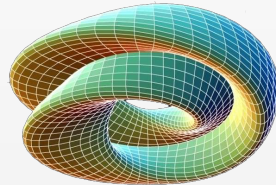
critical phenomena



quantum gravity



string theory



Why are we here? Conformal field theories are hard

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

There are sometimes **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

Typically this happens in presence of a **symmetry**.

We can limit ourselves to a subset of the theory where the associated quantum number Q is large.

Q becomes the **controlling parameter** in a perturbative expansion.

no bootstrap here!



This approach is completely **orthogonal to bootstrap**.

We will use an effective action.

We will access sectors that are **exponentially difficult** to reach with bootstrap.

Summary of the results

Today we will describe a very **concrete example** where this type of simplification happens.

We consider the $O(N)$ **vector model in three dimensions**, which is known to flow in the IR to a **conformal fixed point**.

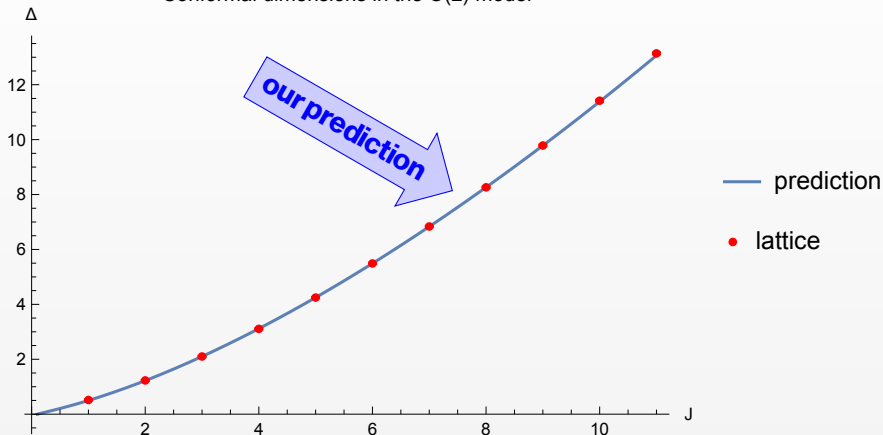
We will find an explicit formula for the **dimension of the lowest-energy state**:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.093 + \mathcal{O}(Q^{-1/2})$$

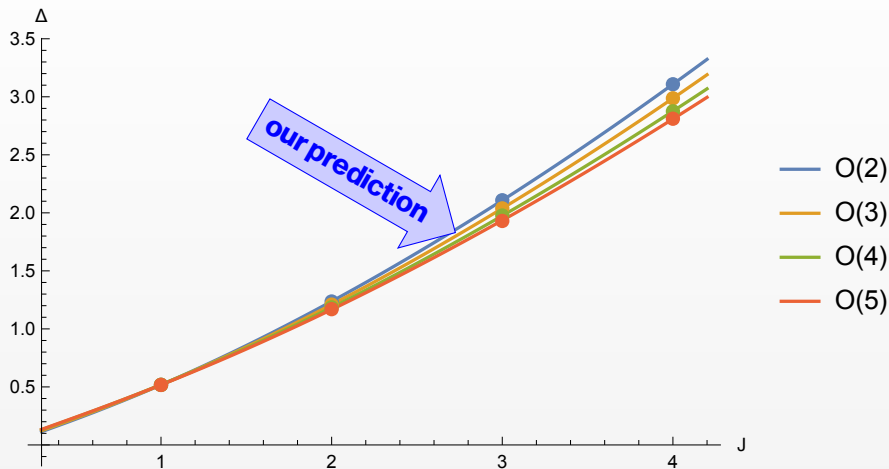
We will see how the very same formula describes the large-R-charge sector of a **supersymmetric $\mathcal{N} = 2, d = 3$ model**.

Summary of the results

Conformal dimensions in the O(2) model



Summary of the results



[Hasenbusch and Vicari]

Scales

We want to write a **Wilsonian effective action**.

Choose a cutoff Λ , separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathcal{D}\phi_H e^{iS(\phi_H, \phi_L)}$$

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too hard

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We need to understand the scales.

- ▶ We look at a finite box of typical length R
- ▶ The $U(1)$ charge Q fixes a **second scale** $\rho^{1/2} \sim Q^{1/2}/R$

We think of the CFT as a Wilsonian effective action at the fixed point with

$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV} = g^2 \Rightarrow Q \gg 1$$

Claim: For $\Lambda \ll \rho^{1/2}$ the **effective action is weakly coupled and under perturbative control** in powers of ρ^{-1} .

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The $O(N)$ model

The UV Lagrangian of the $O(N)$ vector model is of the form

$$\mathcal{L}_{UV} = \partial_\mu \phi^a \partial^\mu \phi^a - g^2 (\phi^a \phi^a)^2,$$

Wilson and Fisher showed that this flows to a conformal IR fixed point.

UV theory $\xrightarrow{\text{RG flow}}$ IR conformal fixed point.

The idea is to make use of this fact to write an **effective Wilsonian action**.

Approximate scale invariance

Consider the $O(2)$ case.

Concretely we set $\phi = a e^{i\chi}$ and we give a **large vacuum expectation value (vev)** to a :

$$\Lambda \ll a^2 \ll g^2.$$

In this limit the Lagrangian is (approximately) **scale-invariant** with corrections $\sim \Lambda/a^2$.

For dimensional reasons it must be

$$\mathcal{L}_{\text{IR}} = \frac{1}{2}(\partial_\mu a)^2 + \frac{b^2}{2}a^2(\partial_\mu \chi)^2 - \frac{R}{8}a^2 - \frac{\lambda}{3}a^6 \\ + (\text{higher derivative terms}).$$

where R is the scalar curvature, and b and λ are numerical constants.

Approximate scale invariance

The charge density is simply

$$\rho := \frac{\delta \mathcal{L}_{\text{IR}}}{\delta \dot{\chi}} = b^2 a^2 \dot{\chi}$$

and using the equations of motion (eom) $a^4 \sim b^2 / \lambda \dot{\chi}^2$ we find that on shell the charge density and its integral are

$$\rho \sim b\sqrt{\lambda} a^4 \quad Q \sim 4\pi R^2 b\sqrt{\lambda} a^4 \quad E_0 \sim \rho^{3/2}$$

so that the condition $\Lambda \ll a^2 \ll g^2$ on the scales becomes (as promised)

$$\frac{1}{R} \ll \Lambda \ll \frac{Q^{1/2}}{R} \ll g^2$$

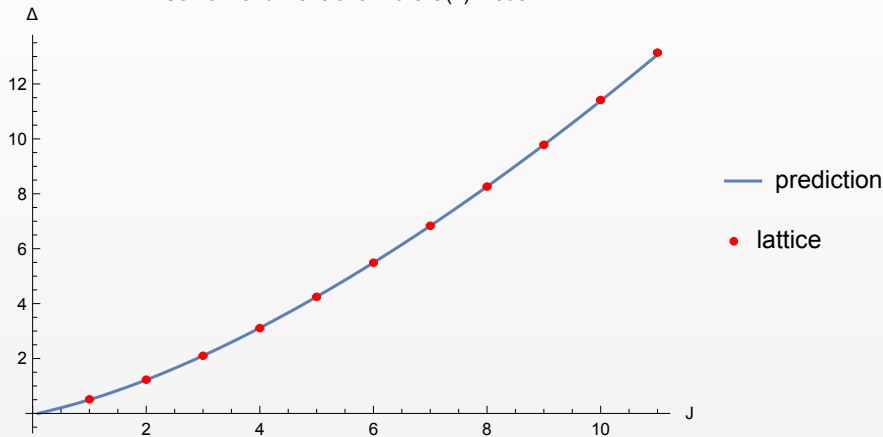
which is consistent if the **charge is large**

$$Q \gg 1.$$

Big charge, small charge

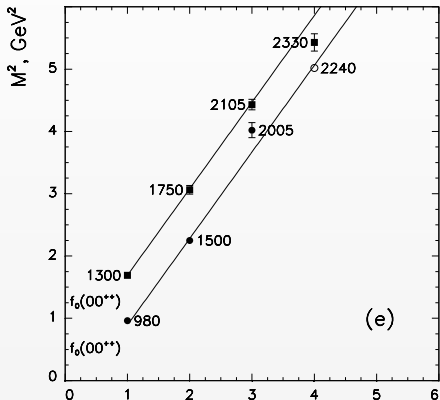
The approximation I am using is supposed to work for large charges. Isn't this a bit too good to be true?

Conformal dimensions in the $O(2)$ model



Big charge, small charge

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Something very similar happens with **Regge trajectories**. The precise prediction is

$$m^2 \propto J \left(1 + \mathcal{O}(J^{-1}) \right)$$

but *experimentally* everything works so well that String Theory was invented from here.

The unreasonable effectiveness of the large quantum number expansion.

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RG analysis

Now I have to justify my claims:

- ▶ Show that the **classical solution** is precisely of the kind found in the previous slide.
- ▶ See how the fluctuations on top of the classical solutions are described by **Goldstone modes**.
- ▶ Show that the higher order terms are **suppressed in $1/Q$** for any value of the couplings b and λ .
- ▶ Derive the **formula for the conformal dimensions**.

P A R E N T A L
A D V I S O R Y
EXPLICIT CONTENT

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Goldstones

Canonical quantization

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Abelian global symmetry at fixed charge

Consider a classical system described by Hamiltonian H with a conserved Abelian global symmetry:

$$\{H, Q\} = 0.$$

we impose the first-class constraint

$$Q = \int \rho \, dx = \bar{Q} = \text{const.}$$

and the corresponding gauge transformation $\delta_\varepsilon f = \{f, \varepsilon Q\}$.
Introduce the canonical conjugate χ to the density ρ

$$\{\chi, Q\} = 1, \quad \text{so that } \delta_\varepsilon \chi = \varepsilon,$$

and assume all the other variables (p_i, q_i) to be gauge invariant.

Abelian global symmetry at fixed charge

For concreteness, consider a **natural Hamiltonian** system:

$$H = \frac{1}{2} \sum_{k=0}^N f_k(q) p_k^2 + \frac{1}{2} \sum_{k=0}^N g_k(q) (\nabla q_k)^2 + V(q).$$

We want to find the ground state of this system.

The Hamiltonian is a **sum of positive terms**, we minimize them separately.

Because of the constraint, $\rho \neq 0$, but we are free to set

$$\nabla q_i = 0, \quad \nabla \chi = 0, \quad p_i = 0, \quad i = 1, \dots, N.$$

Since **nothing depends on the position anymore**, the constraint becomes

$$\int \rho \, dx = \text{vol.} \times \bar{\rho} = \bar{Q}.$$

Abelian global symmetry at fixed charge

The remaining eom are

$$\begin{aligned}\dot{p}_i &= 0, \\ \dot{q}_i &= 0, \\ \dot{\chi} &= f_0(q_i) \bar{\rho}.\end{aligned}$$

They are solved by

$$p_i = 0, \quad q_i = \bar{q}_i(\bar{\rho}), \quad \chi = \mu(\bar{\rho})t,$$

where \bar{q}_i and $\mu(\bar{\rho})$ are constants.

This is the generalization of the classical solution we found in the introduction,

$$a^4 \propto \bar{\rho} \qquad \dot{\chi} \propto \bar{\rho}^{1/2}$$

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Variational description

We want to find a state v that minimizes

$$\langle v|H|v\rangle$$

under the constraints

$$\langle v|v\rangle = 1 \quad \text{and} \quad \langle v|\rho|v\rangle = \bar{\rho}.$$

We introduce the Lagrange multipliers E , m and minimize

$$\langle v|H - E_0 - m\rho|v\rangle.$$

The solution is

$$(H - E_0 - m\rho)|v\rangle = 0.$$

Variational description

To reproduce the classical solution

$$\langle v | \dot{\chi} | v \rangle = \mu ,$$

where μ is the value found earlier. Now

$$\langle v | \dot{\chi} | v \rangle = \langle v | [\chi, H] | v \rangle = m \langle v | [\chi, \rho] | v \rangle ,$$

and since χ, ρ are canonically conjugate, we obtain

$$m = \mu .$$

The quantum Hamiltonian is given by

$$\mathcal{H} = H - \mu \rho - E_0 .$$

μ is now a fixed chemical potential. The vacuum satisfies $\mathcal{H} | v \rangle = 0$.

Goldstones

The chemical potential **breaks explicitly** the symmetry of H from G to $G' \subset G$

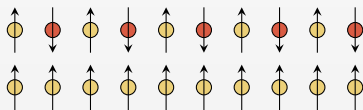
$$\mathcal{H} = H - \mu \rho .$$

The ground state $|v\rangle$ will **break spontaneously** the symmetry to $G'' \subset G'$.

Goldstone tells us: $\dim(G'/G'')$ low energy massless DOF.



We have singled out the time. The system is non-relativistic.



antiferromagnet $\omega \propto p$

ferromagnet $\omega \propto p^2$

A classical vector $O(2n)$ model

Consider the Lagrangian of a $O(2n)$ vector model

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} V(\phi_a \phi_a), \quad a = 1, \dots, 2n,$$

We introduce complex variables

$$\varphi_1 = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2), \dots$$

so the $O(2) \subset O(2n)$ generators act as rotations:

$$\{\varphi_i, \varepsilon Q_j\} = \varepsilon \delta_{ij} \varphi_i.$$

We impose the conditions

$$\int_\Sigma \text{dvol} \rho_i = \bar{Q}_i = V \times \bar{\rho}_i,$$

where the $\bar{\rho}_i$ are fixed.

Ground state

Surprise! In the ground state the system is homogeneous and the solution is given by

$$\varphi_i = \frac{1}{\sqrt{2}} A_i e^{i\mu t}$$

where A_i and μ depend on the fixed charges $\bar{\rho}_i$.

The phase μ is **the same for all fields**, even if all the charges $\bar{\rho}_i$ are different.

We are really fixing **only one** $O(2)$ charge – the values of ρ tell us how this is embedded in the maximal $O(2)^n$ torus.

The classical solution

In the IR the theory becomes conformal (Wilson–Fisher). The Lagrangian is approximately scale invariant and the potential must have the form

$$V(\|\phi\|) = \frac{R}{16}\|\phi\|^2 + \frac{\lambda}{3}\|\phi\|^6,$$

The classical ground state at fixed charge has energy

$$E_{\Sigma}(Q) = \frac{c_{3/2}}{\sqrt{V}}Q^{3/2} + \frac{c_{1/2}}{2}R\sqrt{V}Q^{1/2} + \mathcal{O}(Q^{-1/2}),$$

- ▶ there are two **universal parameters**: $c_{3/2}$ and $c_{1/2}$
- ▶ the result depends on the manifold only via the volume V and the scalar curvature R

How do the **quantum corrections** change this result? How controlled is our approximation?

How many Goldstones?

Using the variational approach, the quantum Hamiltonian is

$$\mathcal{H} = H - \mu(\rho_1 + \rho_2 + \cdots + \rho_k),$$

This breaks the $O(2n)$ symmetry explicitly to $U(n)$. The vacuum

$$\langle \varphi_i \rangle = A_i,$$

breaks $U(n)$ spontaneously to $U(n-1)$ (just rotate the vector $\langle \vec{\varphi} \rangle = (A_1, \dots, A_n)$ into $(0, \dots, 0, v)$).

The dimension of the coset is

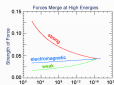
$$\dim G/H = \dim U(n) - \dim U(n-1) = 2n - 1.$$

The system is non-relativistic. This is only an upper bound on the number of Goldstones.

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How many Goldstones?

Pass to the Lagrangian formalism and expand around the classical solution.

$$\varphi_i = e^{i\mu t} \hat{\varphi}_i, \quad i = 1, \dots, n-1$$

$$\varphi_n = \frac{1}{\sqrt{2}} e^{i\mu t + i\hat{\varphi}_{2n}/v} (v + \hat{\varphi}_{2n-1})$$

The (unbroken) $U(n-1)$ symmetry is then realized as $\hat{\varphi}_i \mapsto \tilde{U}_i^j \hat{\varphi}_j$.
The second order Lagrangian becomes:

$$\begin{aligned} \mathcal{L}^{(2)} = & \sum_{i=1}^n (\partial_t - i\mu) \varphi_i^* (\partial_t + i\mu) \varphi_i - \sum_{i=1}^n \nabla \varphi_i^* \nabla \varphi_i \\ & - \sum_{i=1}^n \mu^2 \varphi_i^* \varphi_i - \frac{2c^2}{1-c^2} \mu^2 \phi_{2n-1}^2, \end{aligned}$$

where $\mu^2 = V'(v^2)$ (eom) and $c < 1$ is a dimensionless parameter.

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ask for details

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The massless modes are:

$$\omega_r^2 = c^2 p^2 + \frac{(1 - c^2)^3 p^4}{4\mu^2} + \mathcal{O}(\mu^{-4})$$

$$\omega_{nr}^2 = \frac{p^4}{4\mu^2} - \frac{p^6}{8\mu^4} + \mathcal{O}(\mu^{-6})$$



We have $n - 1$ non-relativistic Goldstones $\omega \propto p^2$ and one relativistic one $\omega \propto p$. The non-relativistic ones are **suppressed at large $\bar{\rho}$** .

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Non-relativistic ones “count double” [Nielsen and Chadha] [Murayama and Watanabe] and we have $2 \times (n - 1) + 1 = 2n - 1 = \dim G/H$.

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Canonical quantization

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Canonical quantization of the non-Abelian sector

The quadratic Hamiltonian in the $\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$ sector

$$H_i = \pi_i^* \pi_i + \nabla \varphi_i^* \nabla \varphi_i + \mu^2 \varphi_i^* \varphi_i - \mu (\pi_i \varphi_i - \pi_i^* \varphi_i^*).$$

Go to Fourier space and expand in terms of canonical operators:

$$\varphi_i(p) = \frac{1}{\sqrt{2\tilde{\omega}(p)}} (a_i(p) + b_i^\dagger(-p)),$$

The Hamiltonian is diagonalized by the choice $\tilde{\omega}^2 = p^2 + \mu^2$:

$$H_i(p) = \left(\sqrt{p^2 + \mu^2} - \mu \right) a_i^\dagger(p) a_i(p) + \left(\sqrt{p^2 + \mu^2} + \mu \right) b_i^\dagger(p) b_i(p).$$

We have **broken Lorentz invariance**, and the symmetry between **particles and antiparticles**.

For $\mu \gg 1$, a is a Goldstone with $\omega \sim \frac{p^2}{2\mu}$ and b is massive.

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Non-relativistic Goldstones

Write the Lagrangian

$$\mathcal{L}_i = (\partial_t - i\mu) \varphi_i^* (\partial_t + i\mu) \varphi_i - \mu^2 \varphi_i^* \varphi_i - \nabla \varphi_i^* \nabla \varphi_i.$$

If $\mu \gg \partial_t$, this is a massless Schrödinger particle:

$$\mathcal{L}_i = i\mu (\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i) - \nabla \varphi_i^* \nabla \varphi_i,$$

The term $\mu(\rho_1 + \dots + \rho_k)$ is a Berry's phase and we get only one classical Goldstone particle instead of two (ferromagnet).

φ and φ^* are canonically conjugate to each other. The Goldstones "count double".

Non-relativistic Goldstones **do not contribute** to the Casimir energy.

The Abelian sector

The Hamiltonian for the $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$ sector (where the mass term appears) is

$$H_n = \frac{1}{2} \left[\pi_{2n-1}^2 + \pi_{2n}^2 + (\nabla \phi_{2n-1})^2 + (\nabla \phi_{2n})^2 + \mu^2 \left(\frac{1+3c^2}{1-c^2} \phi_{2n-1}^2 + \phi_{2n}^2 \right) - \mu (\pi_{2n-1} \phi_{2n} - \pi_{2n} \phi_{2n-1}) \right].$$

Also this can be diagonalized in the oscillators:

$$H_n = c p a_n^\dagger(p) a_n(p) + \frac{2\mu}{\sqrt{1-c^2}} b_n^\dagger(p) b_n(p) + \mathcal{O}\left(\frac{1}{\mu}\right).$$

We see that a is a Goldstone with $\omega = cp$ and b is massive.

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Suppression of the interactions

We have assumed that the quadratic part of the Hamiltonian is the most important and that the rest can be treated as small.

Expand the potential:

$$V(\phi) = V(v^2) + \mu^2 \lambda^{i_1 i_2} \phi_{i_1} \phi_{i_2} + \mu^2 \frac{\lambda^{i_1 i_2 i_3}}{v} \phi_{i_1} \phi_{i_2} \phi_{i_3} + \dots \\ + \mu^2 \frac{\lambda^{i_1 \dots i_m}}{v^{m-2}} \phi_{i_1} \dots \phi_{i_m},$$

where the λ are dimensionless constants and of order $\mathcal{O}(1)$.

To diagonalize H_2 , ϕ_i is of order $\mathcal{O}(\mu^{-1/2})$ so

$$\frac{\mu^2 \lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2}} = \frac{\lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2-2}}.$$

v has the dimensions of a field, $[v] = d/2 - 1$. Overall we have

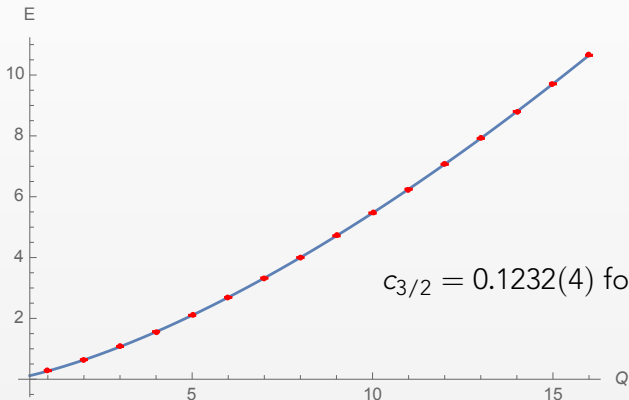
$$\frac{\lambda^{i_1 \dots i_m}}{\mu^{-d+m/2(d-1)}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{(m/2-d/(d-1))}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho} \Omega_m} \quad \Omega_m > 0.$$

Does this work? A small (big) surprise

On a torus $\Sigma = T^2$, the prediction is that the energies go like

$$E_{T^2} = \frac{c_{3/2}}{L} Q^{3/2} + c_0 T^2 + \mathcal{O}(Q^{-1})$$

c_0 is the Casimir energy of our relativistic Goldstone $c_0 = -0.504/L$

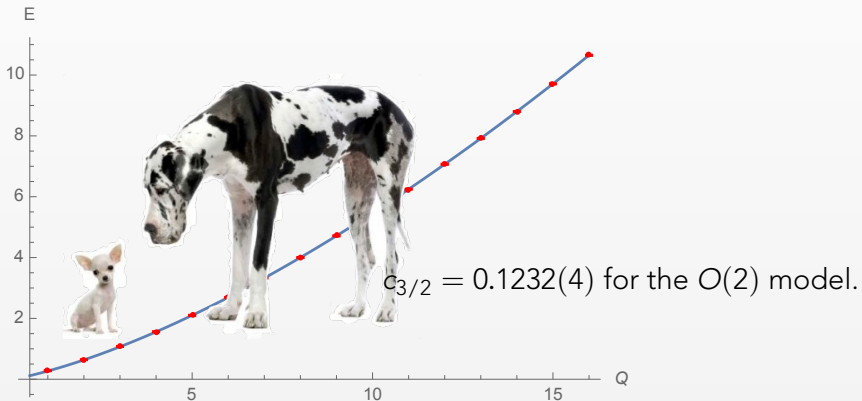


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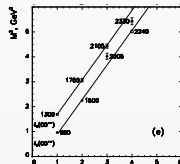
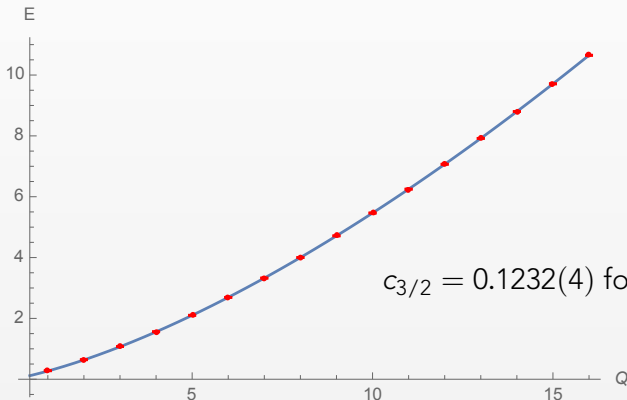


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$c_{3/2} = 0.1232(4)$ for the $O(2)$ model.

The point

- ▶ We started with a generic $O(2n)$ -invariant model
- ▶ Fixing n $U(1)$ charges breaks the symmetry explicitly to $U(n)$. We have a controlling parameter $\bar{\rho}$.
- ▶ The ground state breaks spontaneously to $U(n-1)$
- ▶ There is one relativistic Goldstone (with $c < 1$) and $n-1$ non-relativistic Goldstones, controlled by $\bar{\rho}^{-1}$.
- ▶ We diagonalize the quantum Hamiltonian
- ▶ In the resulting theory, couplings λ in the initial model are suppressed by powers of $\bar{\rho}^{-1}$.
- ▶ In the limit of $\bar{\rho} \rightarrow \infty$, the system is well described by a single Goldstone mode.

Outline

Classical analysis

Goldstones

Canonical quantization

Conformal dimensions

Conclusions

Radial quantization

I have promised to compute the conformal dimensions. Up to this point I have computed energies. How are these related?

We want to describe a conformal theory, so we can start from flat space \mathbb{R}^d and perform a conformal transformation to $\mathbb{R} \times S^{d-1}$:

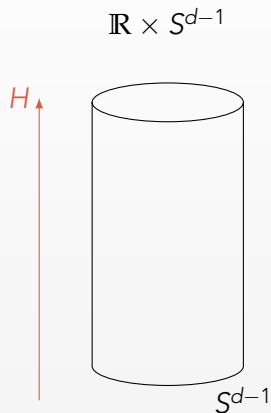
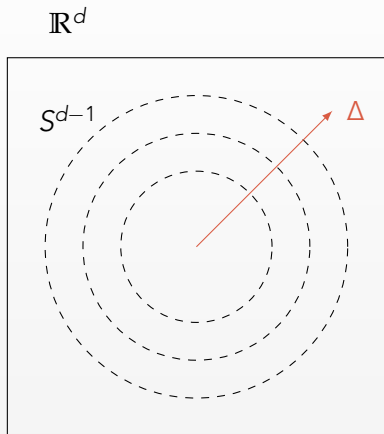
$$ds^2 = d\tau^2 + d\Omega_{d-1}^2 = \frac{1}{r^2} \left(dr^2 + r^2 d\Omega_{d-1}^2 \right),$$

The initial time coordinate has now become the radius r and the Hamiltonian is identified with the dilatation operator.

A state with fixed charge and energy E on $\mathbb{R}_t \times S^{d-1}$ is mapped to an operator on \mathbb{R}^d with conformal dimension

$$\Delta = E.$$

Radial quantization



The action

Up to higher-derivative terms the action must be:

$$S = \frac{1}{2} \int dt d\Omega [g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a - V(\phi^a \phi^a)],$$

where the potential becomes now

$$V(\phi^a \phi^a) = \sum_{a=1}^{2n} \left(\frac{R}{8} (\phi^a)^2 + \frac{\lambda}{3} (\phi^a)^6 \right),$$

R is the Ricci scalar $R = 2$.

Naturalness implies $\lambda = \mathcal{O}(1)$, so **no standard perturbation theory**.

In the limit of **large charge**, we have a single Goldstone mode and the quantum corrections are controlled by $\lambda / \bar{Q}^\# \ll 1$.

Energies

We need just to evaluate the energy of the ground state:

$$E_0 = 4\pi r_0^2 \left(\frac{2\lambda^{1/4}}{3b^{3/2}} \bar{\rho}^{3/2} + \frac{R}{16b^{1/2}\lambda^{1/4}} \sqrt{\bar{\rho}} + \mathcal{O}(\bar{\rho}^{-1/2}) \right).$$

The effect of the Goldstone is of order $\mathcal{O}(Q^0)$ and is the **one-loop vacuum energy**. One just needs to compute a determinant:

$$\log \det \left(-\partial_0^2 + \frac{1}{2} \nabla^2 \right) = \frac{1}{2\sqrt{2}} \sum_{l=0}^{\infty} (2l+1) \sqrt{l(l+1)}$$

which is ζ -function regularized:

$$E_G \simeq \frac{1}{2\sqrt{2}} \left(-\frac{1}{4} - 0.015 \right) = -0.093.$$

This is a **universal prediction** for our construction.

Conformal dimensions

We can put it all together

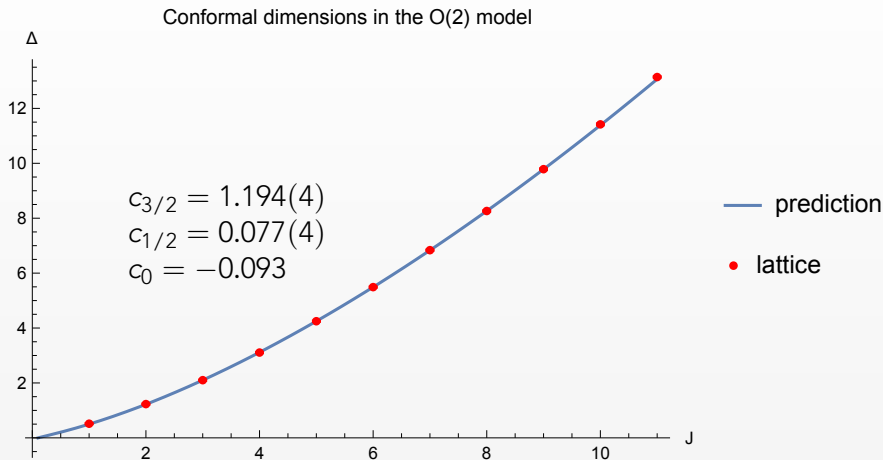
$$\begin{aligned}\Delta_Q &= E_0 + E_G \\ &= \frac{c_{3/2}}{2\sqrt{\pi}} \bar{Q}^{3/2} + 2c_{1/2} \sqrt{\pi} \bar{Q}^{1/2} - 0.093 + \mathcal{O}(\bar{Q}^{-1/2}).\end{aligned}$$

This is a prediction for the conformal dimensions at the Wilson–Fisher point of the $O(n)$ model.

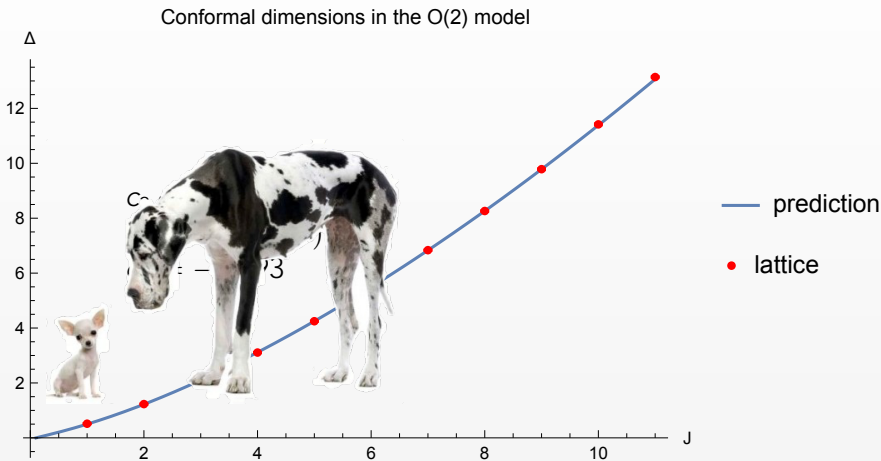
There are two parameters $c_{3/2}$ and $c_{1/2}$ that depend on the details of the model.

They can be computed e.g. on the lattice.

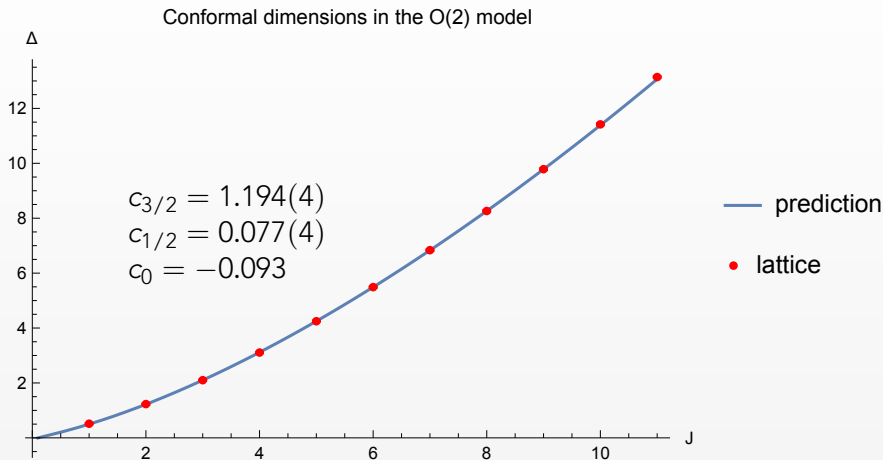
Large charge and the lattice



Large charge and the lattice



Large charge and the lattice



- ▶ $c_{3/2}$ is universal: the same as for the torus energy result;
- ▶ c_0 is NNL order and the computed value is compatible with the measurement.

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Summary of the results

Very concrete examples where a strongly-coupled CFT is simplified in a special sector.

We have considered the $O(N)$ model in three dimensions and seen that in the limit of large $U(1)$ charge Q , the system can be treated perturbatively and non-trivial physical quantities can be calculated.

We have found an explicit formula for the dimension of the lowest-energy state:

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.093$$

The very same formula describes the large- R -charge sector of a supersymmetric $\mathcal{N} = 2, d = 3$ model and dimensions of monopoles in a dual $U(1)$ gauge theory.

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Very concrete examples where a strongly-coupled CFT is simplified in a special sector.

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ask for details

Now what?

- ▶ We would like to get a **better understanding of the $O(n)$ model**. In particular we would like to compute the coefficients $c_{3/2}$ and $c_{1/2}$ from first principles;
- ▶ similarly, we would like to compute these coefficients for the $W = \Phi^3$ model.
- ▶ Why does the approach work numerically for small charge?

We have described a simple example.

We hope our framework is powerful enough to provide **insights in the large- Q behavior of other strongly coupled CFTs** which are in general not tractable with known methods.

*Thank you
for your attention*

Outline

Other systems

The supersymmetric $W = \Phi^3$ model

Monopoles in 3D

The supersymmetric $W = \Phi^3$ model

Consider the $\mathcal{N} = 2$ supersymmetric theory in $D = 3$ with a single chiral superfield Φ , Kähler potential $K = \Phi^\dagger \Phi$ and superpotential $W = 1/3 \Phi^3$.

This theory is well adapted to our formalism:

- ▶ it **flows to** an interacting **superconformal fixed point** [Barnes] [Jafferis]
- ▶ it has **no marginal deformations** or small parameters
- ▶ it has a **continuous global symmetry** (the R-symmetry)

We can compute the dimension of the lowest operator $|Q\rangle$ of charge Q in the limit $Q \gg 1$.

Scale invariance

We choose conventions similar to the $O(2)$ model.

Since $W \sim \Phi^3$, the field has dimension

$$\Phi \propto [\text{mass}]^{2/3}$$

In the IR this means that the Kähler potential goes like $K \propto |\Phi|^{3/2}$ and we fix it to

$$K = \frac{16b_k}{9} |\Phi|^{3/2}$$

so that kinetic term and potential are

$$\mathcal{L}_{\text{kin}} = b_k \frac{\partial \phi \partial \phi}{|\phi|^{1/2}}$$

$$V = \frac{1}{b_k} |\phi|^{9/2}$$

Reduction to Goldstones

At this point everything goes like in the $O(2)$ model: separate absolute value and phase and write the action as

$$\mathcal{L}_{\text{IR}} = \hat{b}_k |\phi|^{3/2} (\partial \chi)^2 + \hat{b}_k \frac{(\partial |\phi|)^2}{|\phi|^{1/2}} + V(|\phi|)$$

+ higher derivatives + fermions

For configurations with $|\phi|$ constant the minimum is for

$$(\partial \chi)^2 \propto |\phi|^3$$

We obtain **precisely the same form** for the action as we had for the Goldstone in the $O(2)$ model.

Reduction to Goldstones

What about the **fermions**? Because of the Yukawa coupling they get a rest energy of order $E \sim |\partial_0 \chi| \sim \rho^{1/2}$: they are **very massive and decouple** from the problem.

We have exactly the same dynamics as we found in the $O(2)$ model. In other words we are in the **same universality class** and the formula for the dimension of the operator Q still stands.

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.093$$

This is somewhat **surprising**: one might have expected $\Delta_Q = Q + \mathcal{O}(Q^0)$ because of supersymmetry.

We find that the states $|Q\rangle$ do not saturate the BPS bound at all: the lowest state in the large- Q sector is **far above the supersymmetric bound!** [Eager].

Monopoles in three dimensions

We are in three dimensions: we can use a **duality transformation to an Abelian theory**.

- ▶ The Noether current maps to the **monopole current**
- ▶ The total Noether charge becomes the **magnetic flux on the sphere**
- ▶ The Noether charge of an operator becomes the **monopole number**

We find that at leading order in the derivative expansion, Weyl-invariance, diffeomorphism covariance, and charge quantization uniquely determine the relation:

$$F_{\mu\nu} = \sqrt{2} |\partial\chi| (*d\chi)_{\mu\nu} = \frac{1}{\sqrt{2}} |\partial\chi| \sqrt{|g|} \varepsilon_{\mu\nu\sigma} \partial^\sigma \chi,$$

Monopoles in three dimensions

The duality means that the effective Lagrangian for the field strength is immediately derived from the leading Goldstone action.

$$\mathcal{L}_{\text{mon}} = b_x |F|^{3/2} + \dots$$

This is consistent with the fact that the Weyl weight of the Lagrangian is 3.

An immediate consequence of the form of the action is that **the dimension of the lowest-lying monopole operator scales as a function of the monopole number to the $\frac{3}{2}$ (for large monopole charge).**