Compensating strong coupling with large charge

Domenico Orlando

Albert Einstein Center for Fundamental Physics – University of Bern

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based on [arXiv:1505.01537], [arXiv:1610.04495] and more to come…

Collaboration with:

L. Alvarez Gaumé (CERN and SCGP), D. Banerjee (DESY), S. Chandrasekharan (Duke), S. Hellerman (IPMU), O. Loukas (AEC Bern), S. Reffert (AEC Bern), M. Watanabe (IPMU).

Outline

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Introduction

Effective action from classical scale invariance

Quantum analysis

Why are we here?

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.
Some of them are just in the middle of coupling-constant space, and all anomalous dimensions are of order 1, and the OPE is just intrinsically complicated, and no parameter of the theory can be dialed to a simplifying limit.

Not all is lost. Even in such cases, there are sometimes sectors of the theory where anomalous dimension and OPE coefficients simplify.

Typically this happens in presence of a symmetry: we can limit ourselves to a subset of the theory where the associated quantum number Q is large.
Q becomes the controlling parameter in a perturbative expansion.

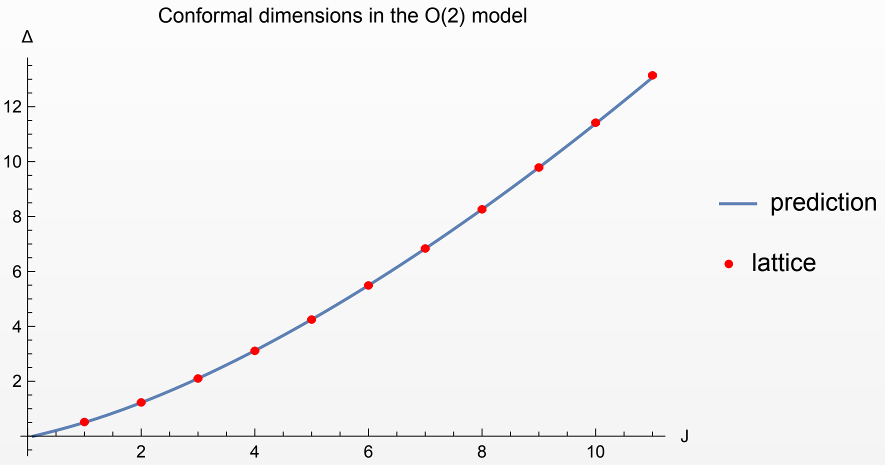
Summary of the results

Today we would like to describe a very concrete example where this type of simplification happens.
We consider the $O(N)$ vector model in three dimensions, which is known to flow in the IR to a conformal fixed point.
We will show that, in the limit of large $U(1)$ charge Q, the system can be treated perturbatively and non-trivial physical quantities can be calculated.
We will find an explicit formula for the dimension of the lowest-energy state:

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.093 + \mathcal{O}\left(Q^{-1/2}\right)$$

We will see how the very same formula describes the large-R-charge sector of a supersymmetric $\mathcal{N} = 2, d = 3$ model.

Summary of the results



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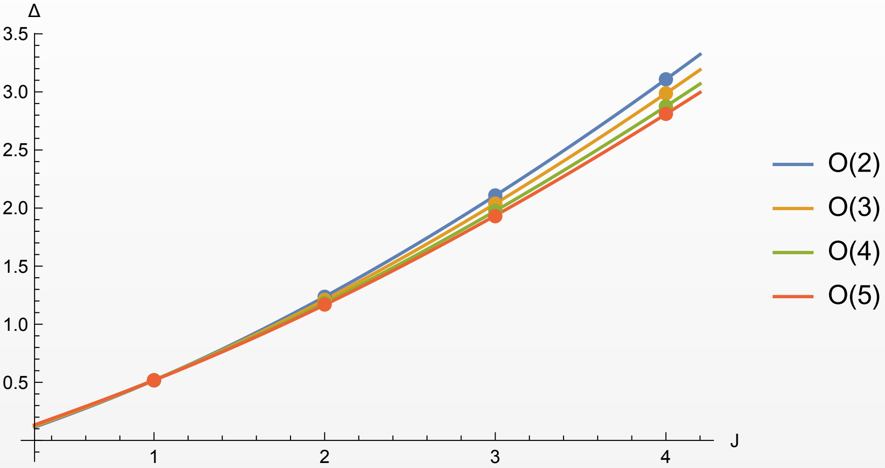


[Hasenbusch and Vicari]

Scales

We want to write a Wilsonian effective action. Choose a cutoff Λ and separate the fields in the path integral into high and low frequency ϕ_H, ϕ_L. Now do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \, e^{iS(\phi_H, \phi_L)}$$

We need to understand the scales.

- We want to compactify on a sphere of radius R
- The $U(1)$ charge density Q fixes a second scale $\rho \sim Q/R^2$

The CFT is a Wilsonian effective action at a fixed point with

$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV} = g^2 \Rightarrow Q \gg 1$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

Outline

The $O(N)$ model

The UV Lagrangian of the $O(N)$ vector model is of the form

$$\mathscr{L}_{\mathrm{UV}} = \partial_\mu \phi^a \partial^\mu \phi^a - g^2 (\phi^a \phi^a)^2,$$

Wilson and Fisher showed that this flows to a conformal IR fixed point.

$$\text{UV theory} \xrightarrow{\text{RG flow}} \text{IR conformal fixed point.}$$

The idea is to make use of this fact to write an effective Wilsonian action.

Approximate scale invariance

For simplicity consider the $O(2)$ case. Concretely we set $\phi = a\, e^{i\chi}$ and we give a large vacuum expectation value (vev) to a:

$$\Lambda \ll a^2 \ll g^2 .$$

In this limit the Lagrangian is (approximately) scale-invariant with corrections $\sim \Lambda / a^2$. It has the form

$$\mathscr{L}_{\mathrm{IR}} = \frac{1}{2}(\partial_\mu a)^2 - f(a)(\partial_\mu \chi)^2 - V(a) + (\text{higher derivative terms}).$$

Here The fields have dimension

$$a \propto [\text{mass}]^{1/2}, \qquad \chi \propto [\text{mass}]^0 .$$

hence

$$\mathscr{L}_{\mathrm{IR}} = \frac{1}{2}(\partial_\mu a)^2 + \frac{b^2}{2} a^2 (\partial_\mu \chi)^2 - \frac{R}{8} a^2 - \frac{\lambda}{3} a^6 + \dots$$

where R is the scalar curvature, and b and λ are numerical constants.

Approximate scale invariance

The charge density is simply

$$\rho := \frac{\delta \mathscr{L}_{\mathrm{IR}}}{\delta \dot{\chi}} = b^2 a^2 \dot{\chi}$$

and using the equations of motion (eom) $a^4 \sim b^2 / \lambda \dot{\chi}^2$ we find that on shell the charge density and its integral are

$$\rho \sim b\sqrt{\lambda}\, a^4 \qquad Q \sim 4\pi R^2 b\sqrt{\lambda}\, a^4 \qquad E_0 \sim \rho^{3/2}$$

so that the condition $\Lambda \ll a^2 \ll g^2$ on the scales becomes (as promised)

$$\frac{1}{R} \ll \Lambda \ll \frac{Q^{1/2}}{R} \ll g^2$$

which is consistent if the charge is large

$$Q \gg 1.$$

Outline

RG analysis

Now I have to justify my claims:

- Show that the classical solution is precisely of the kind found in the previous slide.
- See how the fluctuations on top of the classical solutions are described by Goldstone modes.
- Show that the higher order terms are suppressed in $1/Q$ for any value of the couplings b and λ.
- Derive the formula for the conformal dimensions.

PARENTAL

ADVISORY

EXPLICIT CONTENT

Outline

Outline

Abelian global symmetry at fixed charge

Consider a classical system described by Hamiltonian H with a conserved Abelian global symmetry:

$$\{H, Q\} = 0 .$$

we impose the first-class constraint

$$Q = \int \rho \, dx = \overline{Q} = \text{const} .$$

and the corresponding gauge transformation $\delta_{\varepsilon} f = \{f, \varepsilon Q\}$.
Introduce the canonical conjugate χ to the density ρ

$$\{\chi, Q\} = 1 , \qquad \text{so that } \delta_{\varepsilon} \chi = \varepsilon ,$$

and assume all the other variables (p_i, q_i) to be gauge invariant.

Abelian global symmetry at fixed charge

For concreteness, consider a natural Hamiltonian system:

$$H = \frac{1}{2}\sum_{k=0}^{N} f_k(q)p_k^2 + \frac{1}{2}\sum_{k=0}^{N} g_k(q)(\nabla q_k)^2 + V(q),$$

with $p_0 = \rho$, $q_0 = \chi$ and f_k, g_k functions. We want to find the ground state of this system.

The Hamiltonian is a sum of positive terms, we need to set them each to zero separately.

Because of the constraint, $\rho \neq 0$, but we are free to set

$$\nabla q_i = 0, \quad \nabla \chi = 0, \quad p_i = 0, \quad i = 1, \dots, N.$$

Since nothing depends on the position anymore, the constraint becomes

$$\int \rho \, \mathrm{dx} = \mathrm{vol.} \times \bar{\rho} = \overline{Q}.$$

Abelian global symmetry at fixed charge

The remaining eom are

$$\begin{aligned} \dot{p}_i &= \partial_i f_0 \, \bar{\rho}^2 + \partial_i V = 0 \,, \\ \dot{q}_i &= 0 \,, \\ \dot{\chi} &= f_0(q_i) \, \bar{\rho} \,. \end{aligned}$$

They are solved by

$$p_i = 0 \,, \qquad q_i = \bar{q}_i(\bar{\rho}) \,, \qquad \chi = f_0(\bar{q}_i(\bar{\rho})) \, \bar{\rho} \, t = \mu(\bar{\rho}) t \,,$$

where $\bar{q}_i$ and $\mu(\bar{\rho})$ are constants.

This is the generalization of the classical solution we found in the introduction,

$$a^4 \propto \bar{\rho} \qquad \qquad \dot{\chi} \propto \bar{\rho}^{1/2}$$

Outline

Variational description

We want to find a state v that minimizes

$$\langle v|H|v\rangle$$

under the constraints

$$\langle v|v\rangle = 1 \quad \text{and} \quad \langle v|\,\rho\,|v\rangle = \bar{\rho}\;.$$

We introduce the Lagrange multipliers E, m and minimize

$$\langle v|H - E_0 - m\,\rho\,|v\rangle\;.$$

The solution is

$$(H - E_0 - m\,\rho\,)\,|v\rangle = 0\,.$$

Variational description

To reproduce the classical solution

$$\langle v|\, \dot{\chi}\, |v\rangle = \mu \,,$$

where μ is the value found earlier. Now

$$\langle v|\, \dot{\chi}\, |v\rangle = \langle v|[\chi, H]|v\rangle = m \langle v|[\chi, \rho]|v\rangle \,,$$

and since χ, ρ are canonically conjugate, we obtain

$$m = \mu \,.$$

The quantum Hamiltonian is given by

$$\mathcal{H} = H - \mu\, \rho - E_0 \,.$$

μ is now a fixed chemical potential. The vacuum satisfies $\mathcal{H}\,|v\rangle = 0$.

The Goldstone

The symmetry generated by Q is broken by the ground state $|v\rangle$, so there is a some local operator $A(x)$ such that $\langle v|[Q, A(0)]|v\rangle \neq 0$.

$$\int \mathrm{d}^{d-1}x \, \langle v| e^{i(P\cdot X + Ht)} \rho \; e^{-i(P\cdot X + Ht)} A(0)|v\rangle - \text{h.c.}$$
$$= \sum_p \delta^{(d-1)}(p) \, \langle v| \rho \; \mathrm{e}^{-i(H - \mu Q)t} |p\rangle \; \langle p|A(0)|v\rangle - \text{h.c.} = \text{const.} \neq 0$$

This is possible only if there is a state $|\chi(p)\rangle$ with the property that

$$\lim_{p\to 0} (H - \mu Q) \, |\chi(p)\rangle = 0 \, .$$

This is the Goldstone.

⚠ We have singled out the time. The system is non-relativistic.

A classical vector $O(2n)$ model

Consider the Lagrangian of a $O(2n)$ vector model

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \tfrac{1}{2}V(\phi^a\phi^a), \quad a = 1, \dots, 2n,$$

in $\mathbb{R}_t \times \mathbb{R}^{d-1}$. Now, having in mind that

$$U(n) \subset O(2n),$$

we introduce complex variables

$$\varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \varphi_2 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4), \qquad \dots,$$

so the $U(1) \subset U(n)$ generators act as rotations:

$$\{\varphi_i, \varepsilon Q_j\} = \varepsilon\, \delta_{ij}\varphi_i.$$

Ground state

We impose the conditions

$$\int \mathrm{d}^{d-1}x\, \rho_i = \overline{Q}_i = \mathrm{vol.} \times \bar{\rho}_i \,,$$

where the $\bar{\rho}_i$ are fixed. By the same argument as before, we find that in the ground state, the system is homogeneous and the solution is given by

$$\varphi_i = \frac{1}{\sqrt{2}} A_i e^{i\mu t}$$

where A_i and μ depend on the fixed charges $\bar{\rho}_i$.

$$\bar{\rho}_i = A_i^2 \sqrt{V'(A_1^2 + \cdots + A_n^2)}\,,$$
$$\mu = \sqrt{V'(A_1^2 + \cdots + A_n^2)}\,.$$

The phase μ is the same for all fields, even if all the charges $\bar{\rho}_i$ are different.

How many Goldstones?

Using the variational approach, the quantum Hamiltonian is

$$\mathcal{H} = H - \mu(\rho_1 + \rho_2 + \cdots + \rho_k),$$

This breaks the $O(2n)$ symmetry explicitly to $U(n)$. The vacuum

$$\langle \varphi_i \rangle = A_i,$$

breaks $U(n)$ spontaneously to $U(n-1)$ (just rotate the vector $\langle \vec{\varphi} \rangle = (A_1, \ldots, A_n)$ into $(0, \ldots, 0, v)$).
The dimension of the coset is

$$\dim G/H = \dim U(n) - \dim U(n-1) = 2n - 1.$$

The system is non-relativistic. This is only an upper bound on the number of Goldstones.

How many Goldstones?

Pass to the Lagrangian formalism and expand around the classical solution.

$$\varphi_i = e^{i\mu t}\hat{\varphi}_i, \qquad i = 1, \dots, n-1$$
$$\varphi_n = \frac{1}{\sqrt{2}} e^{i\mu t + i\hat{\phi}_{2n}/v}\left(v + \hat{\phi}_{2n-1}\right)$$

The (unbroken) $U(n-1)$ symmetry is then realized as $\hat{\varphi}_i \mapsto \tilde{U}_i^j \hat{\varphi}_j$. The second order Lagrangian becomes:

$$\mathcal{L}^{(2)} = \sum_{i=1}^{n} (\partial_t - i\mu)\varphi_i^* (\partial_t + i\mu)\varphi_i - \sum_{i=1}^{n} \nabla\varphi_i^* \nabla\varphi_i - \sum_{i=1}^{n} \mu^2 \varphi_i^* \varphi_i - \frac{2c^2}{1-c^2}\mu^2 \phi_{2n-1}^2 ,$$

where $\mu^2 = V'(v^2)$ (eom) and c is a dimensionless parameter.

How many Goldstones?

We have decoupled the problem.
For the first $n-1$ complex fields the inverse propagator is

$$\Delta_i^{-1}(p) = \begin{pmatrix} \frac{1}{2}\left(\omega^2 - p^2\right) & i\omega\,\mu \\ -i\omega\,\mu & \frac{1}{2}\left(\omega^2 - p^2\right) \end{pmatrix},$$

while the n-th field is different because of the mass term for its real component ϕ_{2n-1}:

$$\Delta_k^{-1}(p) = \begin{pmatrix} \omega^2 - p^2 - \frac{4c^2\mu^2}{1-c^2} & 2i\omega\,\mu \\ -2i\omega\,\mu & \omega^2 - p^2 \end{pmatrix}.$$

The dispersion relations of the quasi-particle eigenstates are:

$$\omega = \sqrt{p^2 + \mu^2} \pm \mu \qquad n-1 \text{ times}$$

$$\omega_\pm = \sqrt{p^2 + \frac{2\mu}{1-c^2}\left(\mu \pm \sqrt{\left(1-c^2\right)^2 p^2 + \mu^2}\right)}$$

How many Goldstones?

Expanding for large μ (*i.e.* for large $\bar{\rho}$) we find:

$$\omega^2 = \left(-\mu + \sqrt{p^2 + \mu^2}\right)^2 = \frac{p^4}{4\mu^2} - \frac{p^6}{8\mu^4} + \mathcal{O}\left(\mu^{-6}\right) \qquad n-1 \text{ times}$$

$$\omega^2 = \left(\mu + \sqrt{p^2 + \mu^2}\right)^2 = 4\mu^2 + 2p^2 + \mathcal{O}\left(\mu^{-2}\right) \qquad n-1 \text{ times}$$

$$\omega_-^2 = c^2 p^2 + \frac{\left(1-c^2\right)^3 p^4}{4\mu^2} + \mathcal{O}\left(\mu^{-4}\right) \qquad \text{one time}$$

$$\omega_+^2 = \frac{4\mu^2}{1-c^2} + \left(2-c^2\right)p^2 + \mathcal{O}\left(\mu^{-2}\right) \qquad \text{one time.}$$

We have $n-1$ non-relativistic Goldstones $\omega \propto p^2$ and one relativistic one $\omega \propto p$. The non-relativistic ones are suppressed at large $\bar{\rho}$.

Non-relativistic ones "count double" [Nielsen and Chadha] [Murayama and Watanabe] and we have $2 \times (n-1) + 1 = 2n - 1 = \dim G/H$.

Outline

Canonical quantization of the non-Abelian sector

The quadratic Hamiltonian in the φ_i, $i = 1, \dots, n-1$ is given by

$$H_i = \pi_i^* \pi_i + \nabla \varphi_i^* \nabla \varphi_i + \mu^2 \varphi_i^* \varphi_i - \mu \left(\pi_i \varphi_i - \pi_i^* \varphi_i^* \right).$$

Go to Fourier space and expand in terms of canonical operators:

$$\varphi_i(p) = \frac{1}{\sqrt{2\tilde{\omega}(p)}} \left(a_i(p) + b_i^\dagger(-p) \right),$$

$$\pi_i(p) = -i \sqrt{\frac{\tilde{\omega}(p)}{2}} \left(a_i(p) - b_i^\dagger(-p) \right).$$

The Hamiltonian is diagonalized by the choice $\tilde{\omega}^2 = p^2 + \mu^2$:

$$H_i(p) = \left(\sqrt{p^2 + \mu^2} - \mu \right) a_i^\dagger(p) a_i(p) + \left(\sqrt{p^2 + \mu^2} + \mu \right) b_i^\dagger(p) b_i(p).$$

We have broken Lorentz invariance, and with it the symmetry between particles and antiparticles. For $\mu \gg 1$, a is a non-relativistic Goldstone with $\omega^2 \sim \frac{p^2}{2\mu}$ and b is massive.

Non-relativistic Goldstones

Another way of looking at the problem is to write the Lagrangian

$$\mathcal{L}_i = (\partial_t - i\mu)\varphi_i^*(\partial_t + i\mu)\varphi_i - \mu^2\varphi_i^*\varphi_i - \nabla\varphi_i^*\nabla\varphi_i .$$

If $\mu \gg \partial_t$, the Lagrangian becomes the one of the massless Schrödinger particle:

$$\mathcal{L}_i = i\mu(\dot{\varphi}_i^*\varphi_i - \varphi_i^*\dot{\varphi}_i) - \nabla\varphi_i^*\nabla\varphi_i ,$$

The term $\mu(\rho_1 + \dots + \rho_k)$ acts like a Berry's phase and when it dominates, we get only one classical Goldstone particle instead of two (ferromagnet).

A classical complex field only represents one DOF since φ and φ^* are canonically conjugate to each other. This is why the Goldstones "count double".

The Abelian sector

The Hamiltonian for the Abelian sector (where the mass term appears) is

$$H_n = \frac{1}{2}\Big[\pi_{2n-1}^2 + \pi_{2n}^2 + (\nabla\phi_{2n-1})^2 + (\nabla\phi_{2n})^2 + \mu^2\left(\frac{1+3c^2}{1-c^2}\phi_{2n-1}^2 + \phi_{2n}^2\right) - \mu\left(\pi_{2n-1}\phi_{2n} - \pi_{2n}\phi_{2n-1}\right)\Big].$$

Also this can be diagonalized in the oscillators:

$$\begin{aligned} H_n &= \omega_-(p)a_n^\dagger(p)a_n(p) + \omega_+(p)b_n^\dagger(p)b_n(p) \\ &= c\,p\,a_n^\dagger(p)a_n(p) + \frac{2\mu}{\sqrt{1-c^2}}b_n^\dagger(p)b_n(p) + \mathcal{O}\left(\frac{1}{\mu}\right). \end{aligned}$$

We see that a is a Goldstone and b is massive.

The Abelian sector

The expansion of the fields in oscillators is more complicated.
At large μ we find

$$\phi_{2n-1}(p) \sim \frac{(1-c^2)^{1/4}}{2\sqrt{\mu}} \left(b_n(p) + b_n^\dagger(-p) \right) - \frac{1-c^2}{2c} \frac{p}{\mu} \sqrt{\frac{c}{2p}} \left(a_n(p) + a_n^\dagger(-p) \right),$$

$$\phi_{2n}(p) \sim i\sqrt{\frac{c}{2p}} \left(a_n(p) - a_n^\dagger(-p) \right) + i \frac{(1-c^2)^{3/4}}{2\sqrt{\mu}} \left(b_n(p) - b_n^\dagger(-p) \right).$$

At lowest order, ϕ_{2n} is the Goldstone and ϕ_{2n-1} the massive field.

The Berry's phase term changes the spin wave velocity but does not affect the spectrum qualitatively (antiferromagnet).

Suppression of the interactions

We have assumed that the quadratic part of the Hamiltonian is the most important and that the rest can be treated as small.

At leading order in μ, ϕ_{2k} is the relativistic Goldstone boson. Because of the $O(2n)$ invariance, $V(\phi)$ does not depend on ϕ_{2k}, so the field can appear only in two higher order terms. They are:

$$v\phi_{2k-1}\frac{\phi_{2k}^2}{v^2} \qquad \text{and} \qquad \phi_{2k-1}^2\frac{\phi_{2k}^2}{v^2}.$$

Expanding in oscillators

$$\phi_{2k-1}\frac{\phi_{2k}^2}{v} = \mathcal{O}\left(\frac{1}{v\sqrt{\mu}}\right) \qquad \text{and} \qquad \phi_{2k-1}^2\frac{\phi_{2k}^2}{v^2} = \mathcal{O}\left(\frac{1}{v^2\sqrt{\mu}}\right)$$

They both correct the propagator of the Goldstone by a term $(v^2\mu)^{-1} \ll 1$.

Suppression of the interactions

Now, expanding the potential:

$$V(\phi) = V(v^2) + \mu^2 \lambda^{i_1 i_2} \varphi_{i_1} \varphi_{i_2} + \mu^2 \frac{\lambda^{i_1 i_2 i_3}}{v} \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} + \dots$$
$$+ \mu^2 \frac{\lambda^{i_1 \dots i_m}}{v^{m-2}} \varphi_{i_1} \dots \varphi_{i_m} ,$$

where the λ are dimensionless constants and of order $\mathcal{O}(1)$.
We have seen that φ_i is of order $\mathcal{O}(\mu^{-1/2})$ so the interaction terms among m fields ϕ_i become

$$\frac{\mu^2 \lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2}} = \frac{\lambda^{i_1 \dots i_m}}{v^{m-2} \mu^{m/2-2}} .$$

v has the dimensions of a field, $[v] = d/2 - 1$, so overall we have

$$\frac{\lambda^{i_1 \dots i_m}}{\mu^{-d+m/2(d-1)}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{(m/2-d/(d-1))}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{\Omega_m}} .$$

Suppression of the interactions

$$\frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{(m/2-d/(d-1))}} = \frac{\lambda^{i_1 \dots i_m}}{\bar{\rho}^{\Omega_m}}.$$

- For $m \geq 4$,
$$(d-1)\Omega_m = \frac{m}{2}(d-1) - d > 0$$
and the interactions are suppressed.
- The only dangerous term is $d = 3$, $m = 3$. The cubic term can be either
$$\phi_{2k-1}^3 \qquad \text{or} \qquad \phi_{2k-1}\varphi_i^2$$
they lead to $\mathcal{O}(1)$ corrections to the mass of ϕ_{2k-1}, which is of order $\mathcal{O}(\mu)$.

The point

- We started with a generic $O(2n)$-invariant model
- Fixing n $U(1)$ charges breaks the symmetry explicitly to $U(n)$. We have a controlling parameter $\bar{\rho}$.
- The ground state breaks spontaneously to $U(n-1)$
- There is one relativistic Goldstone (with $c < 1$) and $n-1$ non-relativistic Goldstones, controlled by $\bar{\rho}^{-1}$.
- We diagonalize the quantum Hamiltonian
- In the resulting theory, couplings λ in the initial model are suppressed by powers of $\bar{\rho}^{-1}$.
- In the limit of $\bar{\rho} \to \infty$, the system is well described by a single Goldstone mode.

Outline

Radial quantization

I have promised to compute the conformal dimensions at the Wilson–Fisher point for the $O(n)$ model in three dimensions. Up to this point I have computed energies. How are these related?

We want to describe a conformal theory, so we can start from flat space $\mathbb{R}^d$and perform a conformal transformation to $\mathbb{R} \times S^{d-1}$:

$$ds^2 = d\tau^2 + r_0^2 \, d\Omega_{d-1}^2 = \frac{r_0^2}{r^2}\left(dr^2 + r^2 \, d\Omega_{d-1}^2\right),$$

The initial time coordinate has now become the radius r and the Hamiltonian is identified with the dilatation operator.
A state with fixed charge and energy E on $\mathbb{R}_t \times S^{d-1}$ is mapped to an operator on $\mathbb{R}^d$ with conformal dimension

$$\Delta = r_0 E.$$

The action

We only need the action. We can use large-Q.
At the Wilson–Fisher point the action is approximately scale-invariant. The field ϕ has dimension $1/2$.
Up to higher-derivative terms the action must be:

$$S = \frac{1}{2} \int \mathrm{d}t\, r_0^2\, \mathrm{d}\Omega \left[g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a - V(\phi^a \phi^a) \right],$$

where the potential becomes now

$$V(\phi^a \phi^a) = \sum_{a=1}^{2n} \left(\frac{R}{8} (\phi^a)^2 + \frac{\lambda}{3} (\phi^a)^6 \right),$$

R is the Ricci scalar $R = 2/r_0^2$.
Naturalness implies $\lambda = \mathcal{O}(1)$, so no standard perturbation theory.
In the limit of large charge, we have a single Goldstone mode and the quantum corrections are controlled by $\lambda / \overline{Q}^{1/2} \ll 1$.

Energies

We need just to evaluate the energy of the ground state:

$$E_0 = 4\pi r_0^2 \left(\frac{2\lambda^{1/4}}{3b^{3/2}} \bar{\rho}^{3/2} + \frac{R}{16 b^{1/2} \lambda^{1/4}} \sqrt{\bar{\rho}} + \mathcal{O}\left(\bar{\rho}^{-1/2}\right) \right).$$

The effect of the Goldstone is of order $\mathcal{O}(Q^0)$ and is the one-loop vacuum energy. One just needs to compute a determinant:

$$\log\det\left(-\partial_0^2 + \frac{1}{2}\nabla^2\right) = \frac{1}{2\sqrt{2}} \sum_{l=0}^{\infty} (2l+1)\sqrt{l(l+1)}$$

which is ζ-function regularized:

$$E_G = \frac{1}{2\sqrt{2} r_0}\left(-\frac{1}{4} - 0.015\right).$$

This is a universal prediction for our construction.

Conformal dimensions

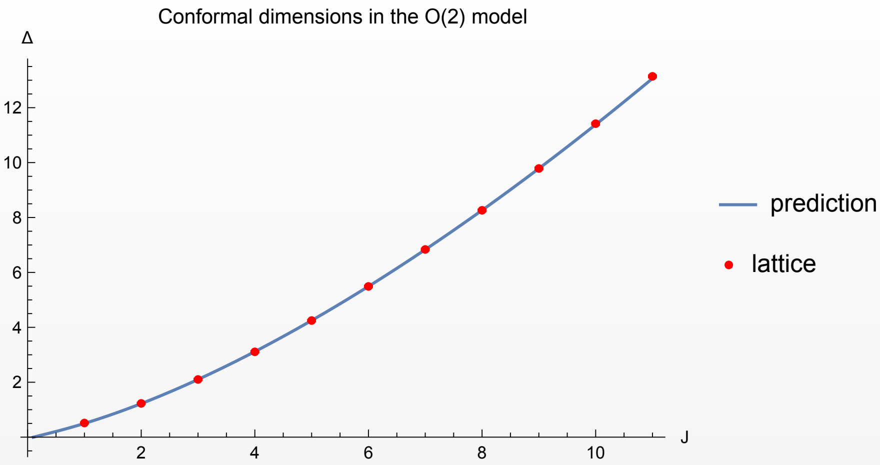
We can put it all together

$$
\begin{aligned}
\Delta_Q &= r_0(E_0 + E_G) \\
&= \frac{\lambda^{1/4}}{3b^{3/2}\sqrt{\pi}}\overline{Q}^{3/2} + \frac{\sqrt{\pi}}{4b^{1/2}\lambda^{1/4}}\overline{Q}^{1/2} - 0.093 + \mathcal{O}\left(\overline{Q}^{-1/2}\right) \\
&= c_{3/2}\overline{Q}^{3/2} + c_{1/2}\overline{Q}^{1/2} - 0.093 + \mathcal{O}\left(\overline{Q}^{-1/2}\right).
\end{aligned}
$$

This is a prediction for the conformal dimensions at the Wilson–Fisher point of the $O(n)$ model.

There are two parameters $c_{3/2}$ and $c_{1/2}$ that depend on the details of the model.
They can be computed *e.g.* on the lattice.

Summary of the results



$$c_{3/2} = 0.3372(7) \qquad \lambda/3 = 2.6(1)$$

$$c_{1/2} = 0.255(3) \qquad b = 0.960(7)$$

Outline

Classical analysis

Goldstones

Canonical quantization

Conformal dimensions

Other systems
- The supersymmetric $W = \Phi^3$ model
- Monopoles in 3D

Conclusions

The supersymmetric $W = \Phi^3$ model

Consider the $\mathcal{N} = 2$ supersymmetric theory in $D = 3$ with a single chiral superfield Φ, Kähler potential $K = \Phi^\dagger \Phi$ and superpotential $W = 1/3\Phi^3$.

This theory is well adapted to our formalism:

- it **flows to** an interacting **superconformal fixed point** [Barnes] [Jafferis]
- it has **no marginal deformations** or small parameters
- it has a **continuous global symmetry** (the R-symmetry)

We can compute the dimension of the lowest operator $|Q\rangle$ of charge Q in the limit $Q \gg 1$.

Scale invariance

We choose conventions similar to the $O(2)$ model.
Since $W \sim \Phi^3$, the field has dimension

$$\Phi \propto [\text{mass}]^{2/3}$$

In the IR this means that the Kähler potential goes like $K \propto |\Phi|^{3/2}$ and we fix it to

$$K = \frac{16 b_k}{9} |\Phi|^{3/2}$$

so that kinetic term and potential are

$$\mathscr{L}_{\text{kin}} = b_k \frac{\partial \phi \, \partial \phi}{|\phi|^{1/2}}$$

$$V = \frac{1}{b_k} |\phi|^{9/2}$$

Reduction to Goldstones

At this point everything goes like in the $O(2)$ model: separate absolute value and phase and write the action as

$$\mathscr{L}_{\mathrm{IR}} = \hat{b}_k |\phi|^{3/2} (\partial\chi)^2 + \hat{b}_k \frac{(\partial|\phi|)^2}{|\phi|^{1/2}} + V(|\phi|) + \text{higher derivatives} + \text{fermions}$$

For configurations with $|\phi|$ constant the minimum is for

$$(\partial\chi)^2 \propto |\phi|^3$$

We obtain precisely the same form for the action as we had for the Goldstone in the $O(2)$ model.

Reduction to Goldstones

What about the **fermions?** Because of the Yukawa coupling they get a rest energy of order $E \sim |\partial_0 \chi| \sim \rho^{1/2}$: they are **very massive and decouple** from the problem.

We have exactly the same dynamics as we found in the $O(2)$ model. In other words we are in the **same universality class** and the formula for the dimension of the operator Q still stands.

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.093$$

This is somewhat **surprising**: one might have expected $\Delta_Q = Q + \mathcal{O}(Q^0)$ because of supersymmetry.
We find that the states $|Q\rangle$ do not saturate the BPS bound at all: the lowest state in the large-Q sector is **far above the supersymmetric bound!** [Eager].

Monopoles in three dimensions

We are in three dimensions: we can use a duality transformation to an Abelian theory.

- The Noether current maps to the monopole current
- The total Noether charge becomes the magnetic flux on the sphere
- The Noether charge of an operator becomes the monopole number

We find that at leading order in the derivative expansion, Weyl-invariance, diffeomorphism covariance, and charge quantization uniquely determine the relation:

$$F_{\mu\nu} = \sqrt{2}|\partial\chi|(*d\chi)_{\mu\nu} = \frac{1}{\sqrt{2}}|\partial\chi|\sqrt{|g|}\,\varepsilon_{\mu\nu\sigma}\partial^{\sigma}\chi\,,$$

Monopoles in three dimensions

The duality means that the effective Lagrangian for the field strength is immediately derived from the leading Goldstone action.

$$\mathscr{L}_{\text{mon}} = b_\chi \left|F\right|^{3/2} + \dots .$$

This is consistent with the fact that the Weyl weight of the Lagrangian is 3.

An immediate consequence of the form of the action is that the dimension of the lowest-lying monopole operator scales ea monopole number to the $\frac{3}{2}$ (for large monopole charge).

Outline

Summary of the results

Today we have seen some very concrete examples where a strongly-coupled CFT is simplified in a special sector.

We have considered the $O(N)$ model in three dimensions and seen that in the limit of large $U(1)$ charge Q, the system can be treated perturbatively and non-trivial physical quantities can be calculated.

We have found an explicit formula for the dimension of the lowest-energy state:

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.093$$

The very same formula describes the large-R-charge sector of a supersymmetric $\mathcal{N} = 2, d = 3$ model.

Now what?

- We would like to get a better understanding of the $O(2)$ model. In particular we would like to compute the coefficients $c_{3/2}$ and $c_{1/2}$ from first principles;
- similarly, we would like to compute these coefficients for the $W = \Phi^3$ model.
- Why does the approach work numerically for small charge?

We have described a simple example.

We hope our framework is powerful enough to provide insights in the large–Q behavior of other strongly coupled CFTs which are in general not tractable with known methods.

Thank you for your attention